The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Schatten-class perturbations of Toeplitz operators

Dominik Schillo

Saarland University

Multivariable Operator Theory, 2017

Joint work with Michael Didas and Jörg Eschmeier (both Saarland University)

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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The classical problem			

Let \mathbb{T} be the unit circle in \mathbb{C} with the canonical probability measure *m*. The *Hardy space with respect to m* will be denoted by

$$H^2(m)=\left\{f\in L^2(m)\;;\;\hat{f}(n)=0\; ext{for all}\;n<0
ight\}\subset L^2(m).$$

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Schatten-class perturbations of Toeplitz operators

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Define

$$H^{\infty}(m) = L^{\infty}(m) \cap H^{2}(m) \subset L^{\infty}(m)$$

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Define

$$H^{\infty}(m) = L^{\infty}(m) \cap H^{2}(m) \subset L^{\infty}(m)$$

and we call

$$I_m = \{ f \in H^\infty(m) ; |f| = 1 m$$
-a.e. $\}$

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the set of inner functions with respect to m.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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The classical problem			

Let $f \in L^{\infty}(m)$. We call

$$T_f \colon H^2(m) \to H^2(m), \ g \mapsto P_{H^2(m)}(fg),$$

where $P_{H^2(m)}$: $L^2(m) \to H^2(m)$ is the orthogonal projection onto $H^2(m)$, the Toeplitz operator with symbol f.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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The classical problem			

Theorem (Brown-Halmos condition I, 1964)

An operator $X \in B(H^2(m))$ is a Toeplitz operator (i.e. there exists a $f \in L^{\infty}(m)$ such that $X = T_f$) if and only if

$$T_z^*XT_z - X = 0,$$

where $z \in L^{\infty}(m)$ is the identity map.

Dominik Schillo Schatten-class perturbations of Toeplitz operators



The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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$$T_z^*XT_z - X = 0,$$

where $z \in L^{\infty}(m)$ is the identity map.

Theorem (Brown-Halmos condition II)

An operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

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for all $u \in I_m$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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The classical problem			

Theorem (Gu, 2004)

An operator $X \in B(H^2(m))$ is a finite rank Toeplitz perturbation (i.e. there exist a $f \in L^{\infty}(m)$ and $F \in F(H^2(m))$ such that $X = T_f + F$) if and only if

$$T_u^*XT_u - X \in F(H^2(m))$$

for all $u \in I_m$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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The classical problem			

Theorem (Xia, 2009)

An operator $X \in B(H^2(m))$ is a compact Toeplitz perturbation (i.e. there exist a $f \in L^{\infty}(m)$ and $K \in K(H^2(m))$ such that $X = T_f + K$) if and only if

$$T_u^*XT_u - X \in K(H^2(m))$$

for all $u \in I_m$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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The classical problem			

Let $p \in [1,\infty)$ and let H be a Hilbert space. An operator $S \in B(H)$ is a *Schatten-p-class* operator if

 $\|S\|_p^p = \operatorname{tr}(|S|^p) < \infty.$



Schatten-class perturbations of Toeplitz operators

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Furthermore, we set

$$\mathcal{S}_p(H) = \left\{ S \in B(H) \; ; \; \left\| S \right\|_p < \infty \right\}$$

equipped with $\|\cdot\|_p$

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Furthermore, we set

$$\mathcal{S}_p(H) = \left\{ S \in B(H) \; ; \; \left\| S \right\|_p < \infty \right\}$$

equipped with $\|\cdot\|_p$ and

 $\mathcal{S}_0(H)=\mathcal{F}(H)$ as well as $\mathcal{S}_\infty(H)=\mathcal{K}(H)$

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both equipped with the operator norm.

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The classical problem 000000●	Formulation of the problem	Main result	Analytic Toeplitz Operators
The classical problem			

Let $\mathbb{T}\subset\mathbb{C}$ be the unit circle with the canonical probability measure $m\in M_1^+(\mathbb{T}).$

Theorem (Gu, Xia, Didas-Eschmeier-S.)

Let $p \in \{0\} \cup [1, \infty]$. An operator $X \in B(H^2(m))$ is a $S_p(H^2(m))$ Toeplitz perturbation (i.e. there exist a $f \in L^{\infty}(m)$ and $S \in S_p(H^2(m))$ such that $X = T_f + S$) if and only if

$$T_u^*XT_u - X \in \mathcal{S}_p(H^2(m))$$

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for all $u \in I_m$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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The classical problem			

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with the canonical probability measure $m \in M_1^+(\mathbb{T})$.

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for all $u \in I_m$.

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The o	lassical problem	Formulation of the problem ●000000	Main result 000000	Analytic Toeplitz Operators 00000
Form	ulation of the problem			
	Definition			
	Let $\mathbb D$ be the u analytic function	init disc in $\mathbb C$ and $\mathcal O(\mathbb D$ ons on $\mathbb D.$) be the set of a	all scalar-valued

Dominik Schillo Schatten-class perturbations of Toeplitz operators



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The classical problem 0000000	Formulation of the problem ●000000	Main result 000000	Analytic Toeplitz Operators 00000	
Formulation of the problem				
Definition				
Let $\mathbb D$ be the	unit disc in $\mathbb C$ and $\mathcal O(\mathbb D)$) be the set of	f all scalar-valued	

analytic functions on $\mathbb D.$ We call

$$A(\mathbb{D}) = ig\{ f \in \mathcal{C}(\overline{\mathbb{D}}) \; ; \; f|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}) ig\}$$

the disc algebra

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Schatten-class perturbations of Toeplitz operators

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The classical problem 0000000	Formulation of the problem ●000000	Main result 000000	Analytic Toeplitz Operators 00000
Formulation of the problem			
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Let \mathbb{D} be the unit disc in \mathbb{C} and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on \mathbb{D} . We call

$$A(\mathbb{D}) = ig\{ f \in C(\overline{\mathbb{D}}) \; ; \; f|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}) ig\}$$

the disc algebra and denote by $\partial_{A(\mathbb{D})}$ the Shilov boundary of $A(\mathbb{D})$ (i.e. the smallest closed subset of $\overline{\mathbb{D}}$ such that

$$\sup_{z\in\overline{\mathbb{D}}}|f(z)|=\sup_{z\in\partial_{\mathcal{A}(\mathbb{D})}}|f(z)|\quad ext{for all }f\in\mathcal{A}(\mathbb{D})).$$

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The classical problem 0000000	Formulation of the problem ●000000	Main result 000000	Analytic Toeplitz Operators 00000
Formulation of the problem			
Definition			

Let \mathbb{D} be the unit disc in \mathbb{C} and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on \mathbb{D} . We call

$${\mathcal A}({\mathbb D})=ig\{f\in {\mathcal C}(\overline{\mathbb D})\;;\; f|_{\mathbb D}\in {\mathcal O}({\mathbb D})ig\}$$

the disc algebra and denote by $\partial_{A(\mathbb{D})}$ the Shilov boundary of $A(\mathbb{D})$ (i.e. the smallest closed subset of $\overline{\mathbb{D}}$ such that

$$\sup_{z\in\overline{\mathbb{D}}}|f(z)|=\sup_{z\in\partial_{\mathcal{A}(\mathbb{D})}}|f(z)|\quad\text{for all }f\in\mathcal{A}(\mathbb{D})).$$

Proposition

We have

$$\partial_{\mathcal{A}(\mathbb{D})} = \partial \mathbb{D} = \mathbb{T}.$$

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Formulation of the problem			

Let $D \subset \mathbb{C}^n$ be a bounded domain. We denote by (i) $A(D) = \{f \in C(\overline{D}) ; f|_D \in \mathcal{O}(D)\} \subset C(\overline{D})$ the domain algebra of D,

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Formulation of the problem			

Let $D \subset \mathbb{C}^n$ be a bounded domain. We denote by

- (i) $A(D) = \{ f \in C(\overline{D}) ; f|_D \in \mathcal{O}(D) \} \subset C(\overline{D})$ the domain algebra of D,
- (ii) $S = \partial_{A(D)}$ the Shilov boundary of A(D) (i.e. the smallest closed subset of \overline{D} such that

$$\sup_{z\in\overline{D}}|f(z)|=\sup_{z\in S}|f(z)|$$

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for all $f \in A(D)$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Formulation of the problem			

Let $D \subset \mathbb{C}^n$ be a stricly pseudoconvex or a bounded symmetric and circled domain. We denote by $\mu \in M_1^+(S)$ the canonical probability measure on S.

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Formulation of the problem			

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Examples

(i)
$$D = \mathbb{B}_n$$
: $S = \partial \mathbb{B}_n$ and $\mu = \sigma$.
(ii) $D = \mathbb{D}^n$: $S = \mathbb{T}^n$ and $\mu = \otimes_n m$.

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The classical problem 0000000	Formulation of the problem 000●000	Main result 000000	Analytic Toeplitz Operators 00000
Formulation of the problem			

We have

$$H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|_{L^2(m)}}}$$
 and $H^{\infty}(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^*}}.$





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Formulation of the problem			

We have

$$H^2(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{\|\cdot\|}_{L^2(m)}} \quad \text{and} \quad H^\infty(m) = \overline{A(\mathbb{D})|_{\mathbb{T}}}^{\tau_{w^*}}$$

Definition

We define

$$H^2(\mu) = \overline{A(D)|_S}^{ au_{\|\cdot\|_{L^2(\mu)}}} \subset L^2(\mu)$$

and

$$H^{\infty}(\mu) = \overline{A(D)|_{S}}^{\tau_{w^{*}}} \subset L^{\infty}(\mu).$$

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Definition

We define

$$\mathsf{H}^2(\mu) = \overline{\mathsf{A}(D)|_{\mathcal{S}}}^{ au_{\|\cdot\|_{L^2(\mu)}}} \subset L^2(\mu)$$

and

$$H^{\infty}(\mu) = \overline{A(D)|_{S}}^{\tau_{w^{*}}} \subset L^{\infty}(\mu).$$

Furthermore, we denote by

$$I_{\mu}=\{f\in H^{\infty}(\mu)\;;\;|f|=1\;\mu ext{-a.e.}\}$$

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Let $f\in L^\infty(\mu).$ We call

$$T_f \colon H^2(\mu) o H^2(\mu), \ g \mapsto P_{H^2(\mu)}(fg),$$

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$$T_f \colon H^2(\mu) \to H^2(\mu), \ g \mapsto P_{H^2(\mu)}(fg),$$

where $P_{H^2(\mu)}: L^2(\mu) \to H^2(\mu)$ is the orthogonal projection onto $H^2(\mu)$, the Toeplitz operator with symbol f.

Theorem (Didas-Eschmeier-Everard, 2011)

An operator $X \in B(H^2(\mu))$ is a Toeplitz operator if and only if

$$T_u^* X T_u - X = 0$$

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for all $u \in I_{\mu}$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Formulation of the problem			

We denote by $H^{\infty}(\mathbb{D}) \subset \mathcal{O}(\mathbb{D})$ the set of all bounded analytic functions on \mathbb{D} .





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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Formulation of the problem			

We denote by $H^{\infty}(\mathbb{D}) \subset \mathcal{O}(\mathbb{D})$ the set of all bounded analytic functions on \mathbb{D} .

Theorem

The map

$$r_m \colon H^{\infty}(\mathbb{D}) \to H^{\infty}(m), \ \theta \mapsto \tau_{w^*} - \lim_{r \to 1} [\theta(r \cdot)] =: \theta^*$$

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is an isometric algebra isomorphism and weak* homeomorphism with $r_m(\theta|_{\mathbb{D}}) = [\theta|_{\mathbb{T}}]$ for all $\theta \in A(\mathbb{D})$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Formulation of the problem			

We denote by $H^{\infty}(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Formulation of the problem			

We denote by $H^{\infty}(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on D.

Theorem

There exists a map

$$r_{\mu} \colon H^{\infty}(D) \to H^{\infty}(\mu), \ \theta \mapsto r_{\mu}(\theta) =: \theta^{*},$$

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which is an isometric algebra isomorphism and weak* homeomorphism with $r_{\mu}(\theta|_D) = [\theta|_S]$ for all $\theta \in A(D)$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex or bounded symmetric and circled domain and $\mu \in M_1^+(S)$ be the probability measure on the Shilov boundary $S = \partial_{A(D)}$ obtained before.

Theorem (Didas-Eschmeier-S., 2017)

Let
$$p \in [1, \infty)$$
. An operator $X \in B(H^2(\mu))$ is a $S_p(H^2(\mu))$
Toeplitz perturbation (i.e. there exist a $f \in L^{\infty}(\mu)$ and
 $S \in S_p(H^2(\mu))$ such that $X = T_f + S$) if and only if

$$T_u^*XT_u - X \in \mathcal{S}_p(H^2(\mu))$$

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for all $u \in I_{\mu}$.

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The classical problem 0000000	Formulation of the problem	Main result ●00000	Analytic Toeplitz Operators 00000
Proof			

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $H^{\infty}(\mu)$ with

$$\tau_{w^*} - \lim_{k \to \infty} \alpha_k = \alpha \in [0, 1)$$

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $H^{\infty}(\mu)$ with

$$\tau_{w^*} - \lim_{k \to \infty} \alpha_k = \alpha \in [0, 1)$$

and $X \in B(H^2(\mu))$ an operator such that

$$Y = au_{\mathsf{WOT}} - \lim_{k o \infty} T^*_{lpha_k} X T_{lpha_k} \in B(H^2(\mu))$$

exists.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $H^{\infty}(\mu)$ with

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$$Y = au_{\mathsf{WOT}} - \lim_{k o \infty} T^*_{lpha_k} X T_{lpha_k} \in B(H^2(\mu))$$

exists. If $T_u^*XT_u - X \in S_{\infty}(H^2(\mu))$ for all $u \in I_{\mu}$, then there exists a function $f \in L^{\infty}(\mu)$ such that

$$X=T_f+\frac{1}{1-\alpha^2}(X-Y).$$

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $H^{\infty}(\mu)$ with

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$$Y = au_{\mathsf{WOT}} - \lim_{k \to \infty} T^*_{\alpha_k} X T_{\alpha_k} \in B(H^2(\mu))$$

exists. If $T_u^*XT_u - X \in S_{\infty}(H^2(\mu))$ for all $u \in I_{\mu}$, then there exists a function $f \in L^{\infty}(\mu)$ such that

$$X = T_f + \frac{1}{1 - \alpha^2} (\boldsymbol{X} - \boldsymbol{Y}).$$

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Proposition (Hiai, 1997)

The map

$$\left\|\cdot\right\|_{p}:\left(B(H^{2}(\mu)), au_{\mathsf{WOT}}
ight)
ightarrow\left[0,\infty
ight],\,\,\mathcal{S}\mapsto\left\|\mathcal{S}
ight\|_{p}$$

is lower semi-continuous.

Dominik Schillo Schatten-class perturbations of Toeplitz operators



The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Proposition (Hiai, 1997)

The map

$$\left\|\cdot\right\|_{p}:\left(B(H^{2}(\mu)), au_{\mathsf{WOT}}
ight)
ightarrow\left[0,\infty
ight],\;S\mapsto\left\|S
ight\|_{p}$$

is lower semi-continuous.

$$\left| \tau_{\mathsf{WOT}^{-}} \lim_{k \to \infty} X - T^*_{\alpha_k} X T_{\alpha_k} \right\|_p \leq \liminf_{k \to \infty} \left\| X - T^*_{\alpha_k} X T_{\alpha_k} \right\|_p$$

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Schatten-class perturbations of Toeplitz operators

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

We denote by

$$I_D = \{ heta \in H^\infty(D) \; ; \; heta^* \in I_\mu\}$$

the set of inner functions with respect to D and μ .

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

We denote by

$$I_D = \{ heta \in H^\infty(D) ; \ heta^* \in I_\mu \}$$

the set of inner functions with respect to D and μ .

Proposition (Aleksandrov, 1984)

Let $\alpha \in [0,1)$. Then there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in I_D such that

$$\tau_{w^*} - \lim_{k \to \infty} \alpha_k^* = \alpha.$$

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Proposition (Xia, 2009)

Let $p \in [1,\infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

$$T^*_u X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_{\mu}$.

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Schatten-class perturbations of Toeplitz operators

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Proposition (Xia, 2009)

Let $p \in [1,\infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

$$T_u^* X T_u - X \in \mathcal{S}_p(H^2(\mu))$$

for all $u \in I_{\mu}$. Then, for all $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\left\| T_{\theta^*}^* X T_{\theta^*} - X \right\|_p \le \varepsilon$$

for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, \mathrm{d}\mu \right| \leq \delta$.

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The classical problem 0000000	Formulation of the problem	Main result 0000●0	Analytic Toeplitz Operators 00000
Proof			

There exists $0 < \delta < 1$ such that $\|T_{\theta^*}^* X T_{\theta^*} - X\|_p \le 1$ for all $\theta \in I_D$ with $\left|\int_S 1 - \theta^* d\mu\right| \le \delta$. Set $\alpha = 1 - \delta/2$.



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Dominik Schillo Schatten-class perturbations of Toeplitz operators

The classical problem 0000000	Formulation of the problem	Main result 0000●0	Analytic Toeplitz Operators 00000
Proof			

There exists $0 < \delta < 1$ such that $||T_{\theta^*}^* X T_{\theta^*} - X||_p \le 1$ for all $\theta \in I_D$ with $|\int_S 1 - \theta^* d\mu| \le \delta$. Set $\alpha = 1 - \delta/2$. \implies There exists $(\alpha_k)_{k \in \mathbb{N}}$ in I_D with τ_{w^*} -lim_{$k \to \infty$} $\alpha_k^* = \alpha$.



Schatten-class perturbations of Toeplitz operators

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

There exists $0 < \delta < 1$ such that $||T^*_{\theta^*}XT_{\theta^*} - X||_p \leq 1$ for all $\theta \in I_D$ with $|\int_S 1 - \theta^* d\mu| \leq \delta$. Set $\alpha = 1 - \delta/2$. \implies There exists $(\alpha_k)_{k \in \mathbb{N}}$ in I_D with τ_{w^*} - $\lim_{k \to \infty} \alpha_k^* = \alpha$. By passing to a subsequence we can achieve that $|\int_D 1 - \alpha_k^* d\mu| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{\mathsf{WOT}} - \lim_{k \to \infty} T^*_{\alpha^*_k} X T_{\alpha^*_k} \in B(H^2(\mu))$$

exists.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

There exists
$$0 < \delta < 1$$
 such that $||T^*_{\theta^*}XT_{\theta^*} - X||_p \leq 1$ for all $\theta \in I_D$ with $|\int_S 1 - \theta^* d\mu| \leq \delta$. Set $\alpha = 1 - \delta/2$.
 \implies There exists $(\alpha_k)_{k \in \mathbb{N}}$ in I_D with $\tau_{w^{*-}} \lim_{k \to \infty} \alpha_k^* = \alpha$.
By passing to a subsequence we can achieve that $|\int_D 1 - \alpha_k^* d\mu| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{\mathsf{WOT}} - \lim_{k \to \infty} T^*_{\alpha^*_k} X T_{\alpha^*_k} \in B(H^2(\mu))$$

exists. Hence

$$\left| \tau_{\mathsf{WOT}^{-}} \lim_{k \to \infty} X - \mathcal{T}_{\alpha_k^*}^* X \mathcal{T}_{\alpha_k^*} \right\|_p \leq \liminf_{k \to \infty} \left\| X - \mathcal{T}_{\alpha_k^*}^* X \mathcal{T}_{\alpha_k^*} \right\|_p \leq 1.$$

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

There exists
$$0 < \delta < 1$$
 such that $||T^*_{\theta^*}XT_{\theta^*} - X||_p \leq 1$ for all $\theta \in I_D$ with $|\int_S 1 - \theta^* d\mu| \leq \delta$. Set $\alpha = 1 - \delta/2$.
 \implies There exists $(\alpha_k)_{k \in \mathbb{N}}$ in I_D with $\tau_{w^{*-}} \lim_{k \to \infty} \alpha_k^* = \alpha$.
By passing to a subsequence we can achieve that $|\int_D 1 - \alpha_k^* d\mu| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{\mathsf{WOT}} - \lim_{k \to \infty} T^*_{\alpha^*_k} X T_{\alpha^*_k} \in B(H^2(\mu))$$

exists. Hence

$$\left\| \tau_{\text{WOT}} \lim_{k \to \infty} X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \le \liminf_{k \to \infty} \left\| X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \le 1.$$
$$\implies X - Y \in \mathcal{S}_p(H^2(\mu)).$$

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Remark

The following ingredients are essential for the proof.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

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Remark

The following ingredients are essential for the proof.

(i) The triple (A(D)|_S, S, μ) is regular (in the sense of Aleksandrov).

The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Proof			

Remark

The following ingredients are essential for the proof.

- (i) The triple (A(D)|_S, S, μ) is regular (in the sense of Aleksandrov).
- (ii) The measure μ is a faithful Henkin measure.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Remark

The following ingredients are essential for the proof.

- (i) The triple (A(D)|_S, S, μ) is regular (in the sense of Aleksandrov).
- (ii) The measure μ is a faithful Henkin measure.

Question

What about $p = 0, \infty$?

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The classical problem 0000000	Formulation of the problem	Main result 000000	Analytic Toeplitz Operators ●0000
Analytic Toeplitz Operator:	s		
Theorem			

An operator $X \in B(H^2(m))$ is an analytic Toeplitz operator (i.e. there exists a $f \in H^{\infty}(m)$ such that $X = T_f$) if and only if

$$[X, T_g] = XT_g - T_g X = 0.$$

for all $g \in H^{\infty}(m)$.

The classical problem 0000000	Formulation of the problem	Main result 000000	Analytic Toeplitz Operators ●0000
Analytic Toeplitz Operators			

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An operator $X \in B(H^2(m))$ is an analytic Toeplitz operator (i.e. there exists a $f \in H^{\infty}(m)$ such that $X = T_f$) if and only if

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for all $g \in H^{\infty}(m)$.

Theorem (Davidson, 1977)

An operator $X \in B(H^2(m))$ has the form $X = T_f + K$ with $f \in H^{\infty}(m) + C(\mathbb{T})$ and $K \in K(H^2(m))$ if and only if

$$XT_g - T_g X \in K(H^2(m)).$$

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for all $g \in H^{\infty}(m)$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Analytic Toeplitz Operators			

Theorem (Hartman, 1958)

We have

$$\mathcal{H}^\infty(m)+\mathcal{C}(\mathbb{T})=\{f\in L^\infty(m)\ ;\ H_f \ is \ compact\}$$

with $H_f = (1 - P_{H^2(m)})M_f|_{H^2(m)}$ for $f \in L^{\infty}(m)$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Analytic Toeplitz Operators			

Theorem (Hartman, 1958)

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$$H^\infty(m)+C(\mathbb{T})=\{f\in L^\infty(m)\ ;\ H_f$$
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with
$$H_f = (1 - P_{H^2(m)})M_f|_{H^2(m)}$$
 for $f \in L^{\infty}(m)$.

Theorem (Davidson, Hartman)

An operator $X \in B(H^2(m))$ has the form $X = T_f + K$ with $f \in L^{\infty}(m)$ such that H_f is compact and $K \in K(H^2(m))$ if and only if

$$XT_g - T_g X \in K(H^2(m)).$$

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for all $g \in H^{\infty}(m)$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Analytic Toeplitz Operators			

 $D = \mathbb{B}_n$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Analytic Toeplitz Operators			

Let
$$n \geq 1$$
 and

$$D = \mathbb{B}_n$$
.

Theorem (Ding-Sun, 1997)

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + K$ for some $f \in L^{\infty}(\mu)$ such that H_f is compact and $K \in K(H^2(\mu))$ if and only if

$$XT_g - T_g X \in K(H^2(\mu))$$

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for all $g \in H^{\infty}(\mu)$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Analytic Toeplitz Operators			

 $D = \mathbb{B}_n$ or $D = \mathbb{D}^n$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Analytic Toeplitz Operators			

$$D = \mathbb{B}_n$$
 or $D = \mathbb{D}^n$.

Theorem (Guo-Wang, 2006)

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + F$ for some $f \in L^{\infty}(\mu)$ such that H_f is of finite rank and $F \in F(H^2(\mu))$ if and only if

$$XT_g - T_g X \in F(H^2(\mu))$$

for all $g \in H^{\infty}(\mu)$.

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The classical problem	Formulation of the problem	Main result	Analytic Toeplitz Operators
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Analytic Toeplitz Operators			

$$D = \mathbb{B}_n$$
 or $D = \mathbb{D}^n$.

Theorem (Didas-Eschmeier-S., 2017)

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + S$ for some $f \in L^{\infty}(\mu)$ such that H_f is in the Schatten-p-class and $S \in S_p(H^2(\mu))$ if and only if

$$XT_g - T_g X \in \mathcal{S}_p(H^2(\mu))$$

for all $g \in H^{\infty}(\mu)$.

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Schatten-class perturbations of Toeplitz operators



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