

Semigroup actions on operator algebras

Evgenios Kakariadis

Newcastle University

Multivariable Operator Theory at the Technion, June 2017



Davidson KR, Fuller AH, Kakariadis ETA, *Semicrossed products of operator algebras by semigroups.*, *Memoirs Amer. Math. Soc.* **247**, No. 1168 (2017), (97 + v pages).



Kakariadis ETA, *On Nica-Pimsner algebras of C^* -dynamical systems over \mathbb{Z}_+^n .*, to appear in *International Mathematics Research Notices*.

I. Framework

In this talk we focus on encoding:

$$\{ \text{C}^*\text{-dynamical systems} \} \iff \{ \text{Operator algebras} \}$$

- Origins: Murray, von Neumann (1936, 1940) – Type I, II, and III factors.
- *C*-crossed products*: are constructed based on a given group action $\alpha: G \rightarrow \text{Aut}(A)$ on a C^* -algebra A by $*$ -automorphisms.
- We turn our focus to semigroup actions $\alpha: P \rightarrow \text{End}(A)$ on a C^* -algebra A by **-endomorphisms*.
- Case example: $P = \mathbb{Z}_+$.

I. Framework

Definition

A *C*-dynamical system* $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$ consists of a *-endomorphism $\alpha: A \rightarrow A$ of a C*-algebra A .

- Use operators to encode the evolution of the system (in discrete time):

$$\begin{array}{cccc} a & \alpha(a) & \alpha^2(a) & \dots \\ | & | & | & \\ t=0 & t=1 & t=2 & \end{array}$$

- The key is to introduce an “external” operator V that satisfies the *covariance relation*

$$a \cdot V = V \cdot \alpha(a) \text{ for all } a \in A.$$

- Sometimes we would like to impose that V is an isometry (so that positive points in time are reversed).

I. Framework

Example of a unitary pair

For $A \subseteq \mathcal{B}(H)$ and $\alpha \in \text{Aut}(A)$ we have a pair (π, U) on $\ell^2(H)$:

U : bilateral shift $\pi(a) = \text{diag}\{\alpha^n(a) : n \in \mathbb{Z}\}$ s.t. $\pi(a)U = U\pi\alpha(a)$.

Notice that U also “undoes” α : $\pi(a) = U\pi\alpha(a)U^*$.

Gauge Invariant Uniqueness Theorem

Every unitary pair (π', U') that admits a gauge action with π' injective lifts to a faithful representation of $C^*(\pi(A), U)$.

- In other words $C^*(\pi(A), U) \simeq A \rtimes \mathbb{Z}$.

Question

What is the analogue for $\alpha \in \text{End}(A)$?

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Cuntz-Pimsner $\mathcal{O}_{(A,\alpha)}$ (Katsura)

Universal C^* -algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \quad \text{for } a \in \ker \alpha^\perp := \{a \in A \mid a \cdot \ker \alpha = (0)\}.$$

Remarks

1. If $\alpha \in \text{Aut}(A)$ then $\ker \alpha^\perp = A$. Thus V is unitary and $\mathcal{O}_{(A,\alpha)} \simeq A \rtimes_\alpha \mathbb{Z}$.
2. When α is injective then $\mathcal{O}(A, \alpha) \simeq A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ for

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & A & \longrightarrow & A_\infty \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha_\infty \\ A & \xrightarrow{\alpha} & A & \longrightarrow & A_\infty \end{array}$$

(Minimal automorphic extension)

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Cuntz-Pimsner $\mathcal{O}_{(A,\alpha)}$ (Katsura)

Universal C*-algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \quad \text{for } a \in \ker \alpha^\perp := \{a \in A \mid a \cdot \ker \alpha = (0)\}.$$

Remarks (Katsura)

3. $\ker \alpha^\perp$ is the largest ideal where the restriction of α is injective.
4. Notice that $a = V\alpha(a)V^*$ for all $a \in \ker \alpha^\perp$.
5. $A \hookrightarrow \mathcal{O}(A, \alpha)$ (Katsura 2004). In fact $\mathcal{O}(A, \alpha)$ is the smallest for which $aV = V\alpha(a)$, V is an isometry and $A \hookrightarrow \mathcal{O}(A, \alpha)$.
6. $\mathcal{O}(A, \alpha)$ satisfies the GIUT.

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Cuntz-Pimsner $\mathcal{O}_{(A,\alpha)}$ (Katsura)

Universal C^* -algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \quad \text{for } a \in \ker \alpha^\perp := \{a \in A \mid a \cdot \ker \alpha = (0)\}.$$

Remarks (Katsura)

7. A is exact if and only if $\mathcal{O}(A, \alpha)$ is exact.
8. If A is nuclear then $\mathcal{O}(A, \alpha)$ is nuclear (but not the converse).
9. If $\mathcal{O}(A, \alpha)$ is nuclear then $A/\ker \alpha^\perp$ is nuclear and $\ker \alpha^\perp \hookrightarrow \mathcal{O}(A, \alpha)^\beta$ is nuclear.

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Cuntz-Pimsner $\mathcal{O}_{(A,\alpha)}$ (Katsura)

Universal C^* -algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \quad \text{for } a \in \ker \alpha^\perp := \{a \in A \mid a \cdot \ker \alpha = (0)\}.$$

Remarks (Katsoulis-Kribs)

10. $\mathcal{O}(A, \alpha)$ is the C^* -envelope in the sense of Arveson for a natural nonselfadjoint operator algebra. Let's pause for a second and put this result into some context.

Intermission: on the C^* -envelope

Operator algebras are subalgebras of $\mathcal{B}(H)$

1. Selfadjoint norm-closed subalgebras, i.e. C^* -algebras.
2. Non-involutive, i.e. *nonselfadjoint operator algebras (nsa)*.

By definition every nsa $\mathcal{A} \subseteq \mathcal{B}(H)$ generates a C^* -algebra $C^*(\mathcal{A})$

It may happen that $\iota_1: \mathcal{A} \rightarrow \mathcal{B}(H_1)$ and $\iota_2: \mathcal{A} \rightarrow \mathcal{B}(H_2)$ but

$$C^*(\iota_1(\mathcal{A})) \not\cong C^*(\iota_2(\mathcal{A})).$$

Example

The disc algebra $\mathbb{A}(\mathbb{D})$ generates the Toeplitz algebra, $C(\overline{\mathbb{D}})$, and $C(\mathbb{T})$. However $C(\mathbb{T})$ is the *minimal C^* -algebra generated by $\mathbb{A}(\mathbb{D})$* , and we call $C(\mathbb{T})$ *the C^* -envelope of $\mathbb{A}(\mathbb{D})$* .

Intermission: on the C^* -envelope

Question, Arveson (1969)

Does every nsa have a C^* -envelope?

Answer: Yes

$\exists \iota: \mathcal{A} \rightarrow \mathcal{B}(H)$ s.t. for any other $\iota': \mathcal{A} \rightarrow \mathcal{B}(K)$, \exists a $*$ -epimorphism $\Phi: C^*(\iota'(\mathcal{A})) \rightarrow C^*(\iota(\mathcal{A}))$ with $\Phi \iota'(a) = \iota(a)$, $\forall a \in \mathcal{A}$.

The $C^*(\iota(\mathcal{A}))$ is the C^* -envelope of \mathcal{A} . We write $C_{\text{env}}^*(\mathcal{A}) = C^*(\iota(\mathcal{A}))$.

Proofs by:

1. Hamana (1979): $C_{\text{env}}^*(\mathcal{A})$ is generated in *the injective envelope*.
2. Ditschel-McCullough (2001): $C_{\text{env}}^*(\mathcal{A})$ is generated by a *maximal dilation*.

Arveson's Program on the C^* -envelope

Determine¹ and examine² the C^* -envelope of a given nsa.

Intermission: on the C^* -envelope

Dilations

Let $T \in \mathcal{B}(H)$. A *power dilation* $U \in \mathcal{B}(K)$ of T is of the form

$$U = \begin{bmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{bmatrix}.$$

A dilation is *maximal* if it has only trivial dilations.

Example

If T is a contraction ($\|T\| \leq 1$), then the maximal dilation is achieved by a unitary U ($U^*U = UU^* = I$).

Dilations

The idea is that by dilating we obtain “better-behaved” objects.

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Going back to $\alpha \in \text{End}(A)$.

Semicrossed product $A \times_\alpha \mathbb{Z}_+$

Universal **nonselafdj** operator algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$ and V is a **contraction**.

Remark

Initiated by Arveson (1967), formally defined by Peters (1984).

Theorem (Muhly-Solel 2006)

*The scp $A \times_\alpha \mathbb{Z}_+$ coincides with the **nsa** generated by A and V such that $a \cdot V = V \cdot \alpha(a)$ and V is an **isometry**.*

Theorem (Katsoulis-Kribs 2005)

The C^ -envelope of $A \times_\alpha \mathbb{Z}_+$ is $\mathcal{O}_{(A,\alpha)}$.*

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Cuntz-Pimsner $\mathcal{O}_{(A,\alpha)}$ (Katsura)

Universal C^* -algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \quad \text{for } a \in \ker \alpha^\perp := \{a \in A \mid a \cdot \ker \alpha = (0)\}.$$

Question

Why such complexity?

Remark

1. Let a faithful $\rho: A \rightarrow \mathcal{B}(H)$ and an isometry V s.t. $\rho(a)V = V\rho\alpha(a)$.
2. If $\rho(a_0) + \sum_{n>0} V_n \rho(a_s) V_n^* = 0$ then $a_0 \in \ker \alpha^\perp$ (Katsura 2004).
3. This happens because such equations transform in $\rho(a_0)(I - VV^*) = 0$.

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Question

Can we connect $\mathcal{O}(A, \alpha)$ to a C^* -crossed product in any case?

Two-step dilation (K. 2011)

1. Add a tail: From (A, α) construct an injective (B, β) given by

$$\begin{array}{ccccccc} \alpha & & & & & & \\ \curvearrowright & & & & & & \\ & A & \xrightarrow{q} & A/\ker \alpha^\perp & \xrightarrow{\text{id}} & A/\ker \alpha^\perp & \xrightarrow{\text{id}} \dots \end{array}$$

2. Extend by a direct limit: For (B, β) form the minimal automorphic extension (B_∞, β_∞) .

II. Operator algebras over $\alpha: \mathbb{Z}_+ \rightarrow \text{End}(A)$

Theorem (K. 2011)

$$\begin{array}{ccc} \alpha: \mathbb{Z}_+ \rightarrow \text{End}(A) & \longrightarrow & \mathcal{O}_{(A,\alpha)} \\ \text{dilation} \downarrow & & \text{strong} \left. \vphantom{\begin{array}{c} \downarrow \\ \downarrow \end{array}} \right\} \text{Morita equivalent} \\ \beta_\infty: \mathbb{Z} \rightarrow \text{Aut}(B_\infty) & \longrightarrow & \mathcal{O}_{(B_\infty,\beta_\infty)} \simeq B_\infty \rtimes_{\beta_\infty} \mathbb{Z} \end{array}$$

Corollary (K. 2011)

Let $A = C(X)$. TFAE:

1. (A, α) is minimal and $\alpha^n \neq \alpha^m$ for all $n, m \in \mathbb{Z}_+$;
2. (B_∞, β_∞) is minimal and $\beta_\infty^n \neq \text{id}$ for all $n \in \mathbb{Z}$ (topol. free);
3. $B_\infty \rtimes_{\beta_\infty} \mathbb{Z}$ is simple;
4. $\mathcal{O}_{(A,\alpha)}$ is simple (has no non-trivial two-sided closed ideals).

III. Program on semigroup actions

Question 1

$$\begin{array}{ccc} \alpha: P \rightarrow \text{End}(A) & \longrightarrow & \text{C}^*\text{-envelope of a scp} \\ \downarrow \text{dilation} & & \downarrow \text{strong} \\ \beta: G \rightarrow \text{Aut}(B) & \longrightarrow & \text{C}^*\text{-crossed product} \end{array} \left. \vphantom{\begin{array}{ccc} \alpha: P \rightarrow \text{End}(A) & \longrightarrow & \text{C}^*\text{-envelope of a scp} \\ \beta: G \rightarrow \text{Aut}(B) & \longrightarrow & \text{C}^*\text{-crossed product} \end{array}} \right\} \text{Morita equivalent ?}$$

Question 2

Is the C*-envelope a Cuntz-type C*-algebra? Can we describe it by *-algebraic relations?

Applications 3

Relate the intrinsic properties of $\alpha: P \rightarrow \text{End}(A)$ to C*-properties of the obtained object.

III. Program on semigroup actions

Davidson-Fuller-K. (2014)

$$\begin{array}{ccc} \alpha: P \rightarrow \text{End}(A) & \longrightarrow & \text{C}^*\text{-envelope of a sem. prod.} \\ \downarrow \text{dilation} & & \text{strong} \left. \vphantom{\begin{array}{c} \text{strong} \\ \text{Morita equivalent} \end{array}} \right\} \text{Morita equivalent} \\ \beta: G \rightarrow \text{Aut}(B) & \longrightarrow & \text{C}^*\text{-crossed product} \end{array}$$

1. We confirm this when P is \mathbb{Z}_+^n , \mathbb{F}_n^+ , a spanning cone, an Ore sgrp.
2. For $P = \mathbb{Z}_+^n$ we identify the Cuntz-Nica-Pimsner algebra.
3. We study the Cuntz-Nica-Pimsner algebras in terms of ideal structure.

K. (2014)

4. We study the Nica-Pimsner algebras in terms of nuclearity, exactness, KMS states.

III. Operator algebras over $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$

Notation

We write $\mathbf{i} = (0, \dots, 0, 1, 0, \dots, 0)$ for all $i = 1, \dots, n$.

Thus $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$ is defined by n commuting $\alpha_{\mathbf{i}} \in \text{End}(A)$.

Requirements

1. n contractions $V_{\mathbf{i}}$ such that $a \cdot V_{\mathbf{i}} = V_{\mathbf{i}} \cdot \alpha_{\mathbf{i}}(a)$.
2. The $V_{\mathbf{i}}$ commute.

Is this enough?

The aim is to reach a crossed product. For $A = \mathbb{C}$ we would like to dilate the $V_{\mathbf{i}}$ to unitaries. Parrott's counterexample shows that this cannot be done for any n .

3. We focus on doubly commuting $V_{\mathbf{i}}$, i.e. $V_{\mathbf{i}}V_{\mathbf{j}}^* = V_{\mathbf{j}}^*V_{\mathbf{i}}$ for $i \neq j$.

III. Operator algebras over $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$

The Nica-covariant semicrossed product $A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n$ (no involution)

Universal **nonselfadjoint** generated by

$$V_s a, \text{ with } a \in A, s \in \mathbb{Z}_+^n,$$

for n doubly commuting **contractions** V_i with $a \cdot V_i = V_i \cdot \alpha_i(a)$.

Remark

A embeds in $A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n$.

Example

For $A \subseteq H$ let $K = H \otimes \ell^2(\mathbb{Z}_+^n)$ and define

$$S_i(\xi \otimes e_s) = \xi \otimes e_{i+s} \text{ and } \pi(a)(\xi \otimes e_s) = \alpha_s(a)\xi \otimes e_s$$

for all $s \in \mathbb{Z}_+^n$ and $\xi \in H$. Then π is a faithful representation of A .

III. Operator algebras over $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$

Question

Why do we call it *Nica* covariant?

Theorem (Davidson-Fuller-K. 2014)

The *Nc-scp* $A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n$ coincides with the **nsa** generated by doubly commuting **isometries** V_i and A such that $a \cdot V_i = V_i \cdot \alpha_i(a)$.

Remark

Doubly commuting isometries form a representation of \mathbb{Z}_+^n in the sense of Nica.

Corollary

If $C_{\text{env}}^*(A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n)$ is generated by A and V then

$$C_{\text{env}}^*(A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n) \simeq \overline{\text{span}}\{V_s a V_t^* : a \in \mathcal{A} \text{ and } s, t \in \mathbb{Z}_+^n\}.$$

III. Reductions

The plan

Dilate a system $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$ to a group action $\beta: \mathbb{Z}^n \rightarrow \text{Aut}(B)$.

Injective case: $\ker \alpha_i = (0)$ for all $i = 1, \dots, n$.

We can then construct the direct limit $\beta_i \in \text{Aut}(B)$ s.t.

$$\begin{array}{ccccc} A_s & \xrightarrow{\alpha_t} & A_{s+t} & \longrightarrow & B \\ \downarrow \alpha_i & & \downarrow \alpha_i & & \downarrow \beta_i \\ A_s & \xrightarrow{\alpha_t} & A_{s+t} & \longrightarrow & B \end{array}$$

where $A_s = A$ for all $s \in \mathbb{Z}_+^2$.

Then $C_{\text{env}}^*(A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n) \simeq B \rtimes_{\beta} \mathbb{Z}^n$ (Corollary Davidson-Fuller-K. 2014).

III. Reductions

The (revised) plan

Dilate a system $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$ where $\ker \alpha_i \neq (0)$ to a system $\beta: \mathbb{Z}_+^n \rightarrow \text{End}(B)$ such that $\ker \beta_i = (0)$.

The $n = 1$ case (K. 2011)

For $I = \ker \alpha^\perp$ let $B = A \oplus c_0(A/I)$ and $\beta(a, (x_n)) = (\alpha(a), a + I, (x_n))$.

$$\begin{array}{ccccccc} & \alpha & & & & & \\ & \curvearrowright & & & & & \\ & A & \xrightarrow{q_I} & A/I & \xrightarrow{\text{id}} & A/I & \xrightarrow{\text{id}} \dots \end{array}$$

The $n = 2$ case

Let $\alpha_1, \alpha_2 \in \text{End}(A)$ such that $\alpha_1 \alpha_2 = \alpha_2 \alpha_1$. We want two injective commuting β_1, β_2 on some $B \supseteq A$ that dilate α_1, α_2 .

III. Non-injective case

A first attempt

Let $I_{(1,1)} := (\ker \alpha_1 \cdot \ker \alpha_2)^\perp$, $I_1 := \bigcap_n \alpha_2^{-n}(I_{(1,1)})$, $I_2 := \bigcap_n \alpha_1^{-n}(I_{(1,1)})$.

Let β_1 be the solid arrows and β_2 the broken arrows:

$$\begin{array}{ccccc}
 & \uparrow & & \uparrow & \\
 \dot{\alpha}_1 \curvearrowright & A/I_2 & \xrightarrow{\dot{q}_1} & A/I_{(1,1)} & \xrightarrow{\text{id}} \dots \\
 & \uparrow & & \uparrow & \\
 \alpha_1 \curvearrowright & A & \xrightarrow{q_1} & A/I_1 & \xrightarrow{\text{id}} \dots \\
 & \uparrow & & \uparrow & \\
 & \alpha_2 \curvearrowright & & \alpha_2 \curvearrowright &
 \end{array}$$

with $\dot{\alpha}_1 q_2 = q_1 \alpha_2$ and $\dot{q}_1 q_1 = q_{(1,1)}$ (plus the symmetrical ones).

Then β is injective and generalises the $n = 1$ case.

However this construction is bound to fail!

III. Non-injective case

How did we end up with $I_{(1,1)} = (\ker \alpha_1 \cdot \ker \alpha_2)^\perp$?

1. Let a faithful $\rho: A \rightarrow \mathcal{B}(H)$ and doubly commuting isometries V_i such that

$$\rho(a)V_i = V_i\rho\alpha_i(a).$$

2. Because of a gauge action, we will have to deal with equations

$$\rho(a_0) + \sum_{s>0} V_s\rho(a_s)V_s^* = 0.$$

3. This transforms into

$$\rho(a_0)(I - V_1V_1^*)(I - V_2V_2^*) = 0.$$

4. From this we get that $a_0 \perp \ker \alpha_1, \ker \alpha_2$.

III. Non-injective case

Why isn't $I_{(1,1)} = (\ker \alpha_1 \cdot \ker \alpha_2)^\perp$ enough?

However we will also have equations of the form

$$\rho(a_0) + \sum_{n>0} V_{(n,0)} \rho(a_n) V_{(n,0)}^* = 0$$

which transform into

$$\rho(a_0)(I - V_1 V_1^*) = 0.$$

From this we get that $a_0 \perp \ker \alpha_1$.

From this we also get that $\alpha_{(0,n)}(a_0) \perp \ker \alpha_1$ for all $n > 0$.

This happens because $\rho \alpha_2(a) = V_2^* \rho(a) V_2$.

So we need the ideal $I_1 = \bigcap_n \alpha_2^{-n}(\ker \alpha_1^\perp)$ instead of $\bigcap_n \alpha_2^{-n}(I)$.

And of course its symmetrical I_2 .

III. Non-injective case

Correct tail

$$I_{(1,1)} = (\ker \alpha_1 \cdot \ker \alpha_2)^\perp \quad I_1 = \bigcap_n \alpha_2^{-n}(\ker \alpha_1^\perp) \quad I_2 = \bigcap_n \alpha_1^{-n}(\ker \alpha_2^\perp).$$

Then define β_1 and β_2 by

$$\begin{array}{ccccc}
 & \uparrow & & \uparrow & \\
 & \text{id} & & \text{id} & \\
 \dot{\alpha}_1 \curvearrowright & A/I_2 & \xrightarrow{\dot{q}_1} & A/I_{(1,1)} & \xrightarrow{\text{id}} \dots \\
 & \uparrow & & \uparrow & \\
 \alpha_1 \curvearrowright & A & \xrightarrow{q_1} & A/I_1 & \xrightarrow{\text{id}} \dots \\
 & \uparrow & & \uparrow & \\
 & \alpha_2 \swarrow \searrow & & \alpha_2 \swarrow \searrow & \\
 & & & &
 \end{array}$$

with $\dot{\alpha}_1 q_2 = q_1 \alpha_2$ and $\dot{q}_2 q_1 = q_{(1,1)}$ (plus the symmetrical ones).

Then β_1 and β_2 generalise the $n = 1$ case.

It is not immediate but they are commuting and injective.

III. General construction

For $\underline{x} = (x_1, \dots, x_n) \in \mathbb{Z}_+^n$, define

$$\text{supp}(\underline{x}) = \{\mathbf{i} : x_i \neq 0\} \text{ and } \underline{x}^\perp = \{\underline{y} \in \mathbb{Z}_+^n : \text{supp}(\underline{y}) \cap \text{supp}(\underline{x}) = \emptyset\}$$

and let the ideals

$$I_{\underline{x}} = \bigcap_{\underline{y} \in \underline{x}^\perp} \alpha_{\underline{y}}^{-1} \left(\left(\bigcap_{\mathbf{i} \in \text{supp}(\underline{x})} \ker \alpha_{\mathbf{i}} \right)^\perp \right).$$

Let $B_{\underline{x}} = A/I_{\underline{x}}$ and on the C*-algebra

$$B = \sum_{\underline{x} \in \mathbb{Z}_+^n}^\oplus B_{\underline{x}}$$

define the *-endomorphisms

$$\beta_{\mathbf{i}}(q_{\underline{x}}(a) \otimes e_{\underline{x}}) = \begin{cases} q_{\underline{x}} \alpha_{\mathbf{i}}(a) \otimes e_{\underline{x}} + q_{\underline{x}+\mathbf{i}}(a) \otimes e_{\underline{x}+\mathbf{i}} & \text{for } \mathbf{i} \in \underline{x}^\perp, \\ q_{\underline{x}}(a) \otimes e_{\underline{x}+\mathbf{i}} & \text{for } \mathbf{i} \in \text{supp}(\underline{x}). \end{cases}$$

Then the $\beta_{\mathbf{i}}$ commute and are injective (this is not trivial).

III. C*-envelope

Theorem (Davidson-Fuller-K. 2014)

Let $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$ be a semigroup action. Apply the constructions:

1. dilate (A, α) to the injective (B, β) by adding a tail;
2. use the direct limit to extend it to $\beta_\infty: \mathbb{Z}^n \rightarrow \text{Aut}(B_\infty)$.

Then the C*-envelope of $A \times_\alpha^{\text{nc}} \mathbb{Z}_+^n$ is Morita equivalent to $B_\infty \rtimes_{\beta_\infty} \mathbb{Z}^n$.

Remarks

1. The C*-envelope is defined by a co-universal property for which

$$C_{\text{env}}^*(A \times_\alpha^{\text{nc}} \mathbb{Z}_+^n) \simeq \overline{\text{span}}\{V_s a V_t^* : a \in \mathcal{A} \text{ and } s, t \in \mathbb{Z}_+^n\}.$$

2. This was one of the challenging points in the proof.

What about the structure of the C*-envelope?

Can we identify the C*-envelope by C*-algebraic relations?

III. Towards a Cuntz algebra

Recall

For $n = 2$ we arrived to the equalities

1. $a(I - V_1 V_1^*) = 0$;
2. $a(I - V_2 V_2^*) = 0$;
3. $a(I - V_1 V_1^*)(I - V_2 V_2^*) = 0$;

subject to a . Then we used the solutions/ideals to produce the tail. This appears to be more than an innocent coincidence.

The Cuntz-Nica-Pimsner algebra for $n = 2$ case

It is the universal C^* -algebra such that: (a) V_i are doubly commuting isometries; (b) $aV_i = V_i\alpha_i(a)$; and (c) we have

- c.1 $a(I - V_1 V_1^*) = 0$ for all $a \in \bigcap_n \alpha_2^{-n}(\ker \alpha_1^\perp)$;
- c.2 $a(I - V_2 V_2^*) = 0$ for all $a \in \bigcap_n \alpha_1^{-n}(\ker \alpha_2^\perp)$;
- c.3 $a(I - V_1 V_1^*)(I - V_2 V_2^*) = 0$ for all $a \in (\ker \alpha_1 \cdot \ker \alpha_2)^\perp$.

III. The Cuntz-Nica-Pimsner algebra

Definition (Davidson-Fuller-K. 2014)

The *Cuntz-Nica-Pimsner algebra* $\mathcal{NO}(A, \alpha)$ of $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$ is the universal C^* -algebra generated by A and V_i so that:

1. V_i are commuting isometries;
2. $aV_i = V_i\alpha_i(a)$; and
3. $a \cdot \prod_{i \in \text{supp}(\underline{x})} (I - V_i V_i^*) = 0$ for $a \in \bigcap_{\underline{y} \in \underline{x}^\perp} \alpha_{\underline{y}}^{-1} \left(\left(\bigcap_{i \in \text{supp}(\underline{x})} \ker \alpha_i \right)^\perp \right)$.

Corollary (Davidson-Fuller-K. 2014)

1. The C^* -envelope of $A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n$ is $\mathcal{NO}(A, \alpha)$.
2. For $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$ there exists a dilation $\beta_\infty: \mathbb{Z}^n \rightarrow \text{Aut}(B_\infty)$ such that $\mathcal{NO}(A, \alpha) \stackrel{\text{sMe}}{\sim} B_\infty \rtimes_{\beta_\infty} \mathbb{Z}^n$.
3. $\mathcal{NO}(A, \alpha)$ satisfies the GIUT with respect to the CNP representations.

III. Simplicity

Theorem (Davidson-Fuller-K. 2014)

$$\begin{array}{ccc}
 \alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A) & \longrightarrow & \mathcal{NO}(A, \alpha) \simeq \mathbf{C}_{\text{env}}^*(A \times_{\alpha}^{\text{nc}} \mathbb{Z}_+^n) \\
 \downarrow \text{dilation} & & \downarrow \text{strong} \\
 \beta_{\infty}: \mathbb{Z}^n \rightarrow \text{Aut}(B_{\infty}) & \longrightarrow & B_{\infty} \rtimes_{\beta_{\infty}} \mathbb{Z}^n
 \end{array}
 \left. \vphantom{\begin{array}{ccc} \alpha & & \mathcal{NO} \\ \downarrow & & \downarrow \\ \beta_{\infty} & & B_{\infty} \rtimes_{\beta_{\infty}} \mathbb{Z}^n \end{array}} \right\} \text{Morita equivalent}$$

Corollary (Davidson-Fuller-K. 2014)

Let $A = C(X)$ and let $\phi_s: X \rightarrow X$ related to $\alpha_s: X \rightarrow X$. TFAE:

1. (A, α) is minimal and $\{x \in X \mid \phi_s(x) \neq \phi_r(x)\}^{\circ} = \emptyset$ for all $s, r \in \mathbb{Z}_+^n$ (top. free);
2. $(B_{\infty}, \beta_{\infty})$ is minimal and topologically free;
3. $B_{\infty} \rtimes_{\beta_{\infty}} \mathbb{Z}$ is simple;
4. $\mathcal{NO}(A, \alpha)$ is simple.

III. Exactness/Nuclearity

Theorem (K. 2014)

$\mathcal{NO}(A, \alpha)$ is exact if and only if A is exact.

Theorem (K. 2014)

Let $\beta_\infty: \mathbb{Z}^n \rightarrow \text{Aut}(B_\infty)$ be the automorphic dilation of $\alpha: \mathbb{Z}_+^n \rightarrow \text{End}(A)$.
TFAE:

1. the embeddings $A, A/I_s \hookrightarrow B_\infty$ are nuclear for all $s \in \mathbb{Z}_+^n$;
2. B_∞ is nuclear;
3. $B_\infty \rtimes_{\beta_\infty} \mathbb{Z}^n$ is nuclear;
4. $\mathcal{NO}(A, \alpha)$ is nuclear.

Proposition (K. 2014)

If A is nuclear or if $A \hookrightarrow C^*(V_{n1} a V_{n1}^* \mid a \in A, n \in \mathbb{Z}_+)$ is nuclear then $\mathcal{NO}(A, \alpha)$ is nuclear. The converse is not true.

IV. Remarks

Remarks on $\mathcal{NT}(A, \alpha)$ (K. 2014)

1. There is a second variant, the Toeplitz-Nica-Pimsner algebra.
2. For this we get A is nuclear (resp. exact) if and only if $\mathcal{NT}(A, \alpha)$ is nuclear (resp. exact).

KMS states (K. 2014)

3. The gauge action implements an action of \mathbb{R} on the Nica-Pimsner algebras. We are able to identify all KMS states at finite temperature: for any $T < \infty$ there is exactly one $\text{KMS}_{1/T}$ state.
4. For $T = \infty$ the KMS states are the tracial states and there is no bijection (there might be more than one).

IV. Remarks

Remarks on simplicity

5. Kalantar-Kennedy show that simplicity of the reduced C^* -crossed product is equivalent to topological freeness of the group action on a boundary.
6. We are working towards formulating this property for semigroups and showing its stability under the automorphic dilation.

Remarks on product systems

7. Both $\mathcal{NT}(A, \alpha)$ and $\mathcal{NO}(A, \alpha)$ are examples of C^* -algebras associated to product systems.
8. A gauge invariance uniqueness theorem for general Toeplitz-Nica-Pimsner algebras is easy to obtain by our methods.
9. We believe that the same is true for the Cuntz-Nica-Pimsner algebras (with Adam Dor-On).

The end.