Matrix Convex Sets and Dilations

Ben Passer, joint with Orr Shalit and Baruch Solel

Technion-Israel Institute of Technology

2017 MVOT Conference
for Baruch Solel's 65th Birthday

Compressions and Dilations

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$N = (N_1, \ldots, N_d) \in B(\mathcal{K})^d$

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We want to start with generic/bad X and reach a "pleasant" dilation N.

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Theorem (Sz.-Nagy)

If $T \in B(\mathcal{H})$ is a contraction, then there is an isometry $V : \mathcal{H} \to \mathcal{K}$ and a unitary $U \in B(\mathcal{K})$ such that

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Theorem (Ando)

For any pair of commuting contractions $T_1, T_2 \in B(\mathcal{H})$, there exist a pair of commuting unitaries $U_1, U_2 \in B(\mathcal{K})$ and an isometry $V : \mathcal{H} \to \mathcal{K}$ with

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Theorem (Halmos)

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The dimension of matrices is fixed at $n \times n$, but the number of matrices d is NOT fixed.

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This implies that $\mathcal{S}$ is an nc set (plus, use $k = 1$ to get simultaneous unitary conjugations), and that each level $\mathcal{S}_n$ is convex (use $V_i = \sqrt{t_i}\, I_n$).

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Study the minimal and maximal matrix convex sets with ground level K. We assume K is compact.

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Suppose that $K \subseteq \mathbb{R}^d$ where K has nice symmetry or invariance properties. Then

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Explanations

This slide should be skipped unless someone asks a question.

Symmetry/invariance properties

There exist k real $d \times d$ matrices $\lambda^{(1)}, \dots, \lambda^{(k)}$ of rank one such that $I_d \in \text{conv}\{\lambda^{(1)}, \dots, \lambda^{(k)}\}$ and

$$\lambda^{(m)} K \subseteq d \cdot K \quad , \quad m = 1, \dots, k$$

e.g., invariant under permutations and sign changes of coordinates.
or more generally: invariant under projections onto orthonormal basis.

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4. Computed some examples of minimal dilation hulls, and made some general conclusions about minimal dilation hulls using the above.

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For $a_1, \ldots, a_d > 0$*,*

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Invertible affine transformations T factor through $\mathcal{W}^{\max}$ and $\mathcal{W}^{\min}$.

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The ball tells a different story:

Dilating a ball to a ball

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There exists a tuple $(F_1, \ldots, F_d)$ of pairwise anticommuting, self-adjoint, unitary, $2^{d-1} \times 2^{d-1}$ matrices such that for any $(y_1, \ldots, y_d) \in \mathbb{R}^d$,

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It is easy to add in a shift and scale of the ball $\overline{\mathbb{B}}_d^2$ on the left side, too.

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Is any circumscribing simplex of K a minimal dilation hull of K? (We don't even know this when K is the ball!)

A Special Case for Circumscribing Simplices

Definition

$K \subset \mathbb{R}^d$ is simplex-pointed at x if $x \in K$ and there is an open set $O \subseteq \mathbb{R}^d$ such that $x \in O$ and $\overline{O \cap K}$ is a d-simplex.

Theorem

Suppose that K is simplex-pointed at x, and Δ is a simplex containing K. If x is a vertex of Δ, the edges of Δ based at x point in the same direction as those of $\overline{O \cap K}$, and there is a point $y \in K$ in the interior of the face F of Δ which excludes x, then Δ is a minimal dilation hull of K.

This is a ridiculously specific example of a circumscribing simplex, but it occurs at least once in nature for each $p \geq 1$:

Corollary

Let $\overline{\mathbb{B}}^d_{p,+}$ denote the positive section of the ℓ^p ball in $\mathbb{R}^d$. Then $d^{1-1/p} \cdot \overline{\mathbb{B}}^d_{1,+}$ is a minimal dilation hull of $\overline{\mathbb{B}}^d_{p,+}$. Further, $\theta(\overline{\mathbb{B}}^d_{p,+}) = d^{1-1/p}$.

Thank you!

(2 Bonus Slides follow - these were not used in the actual talk)

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Main idea of proof: use an affine transformation to talk about positive operators, and manipulate dilations of projections with disjoint ranges.

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Many of the estimates involving $\mathcal{W}^{\max}(\overline{\mathbb{B}}_2^d)$ reduce to the case of these $2^{d-1} \times 2^{d-1}$ self-adjoint pairwise anticommuting unitaries $F_1, \ldots, F_d$ (Pauli matrices).

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Other relations? Other finitely presented universal C^*-algebras?