A Complex Harmonic Analysis Approach to the Geometric Arveson-Douglas Conjecture

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Hilbert Submodule

H_n^2 the Drury-Arveson space, or symmetric Fock space
I an ideal in $\mathbb{C}[z_1, \ldots, z_n]$, $[I]$ its closure of I in H_n^2
$\mathbb{C}[z_1, \ldots, z_n]$ acts on $[I]$ by

$$p \in \mathbb{C}[z_1, \ldots, z_n] \mapsto R_p = M_p|_{[I]}.$$

$[I]$ becomes a Hilbert submodule of H_n^2.

Quotient Module

$Q = [I]^{\perp} \cong H_n^2/[I]$, Q becomes a quotient module

Module action

$$p \mapsto S_p = QM_p|_Q$$

where Q also denotes the projection operator onto the quotient module Q.

Arveson-Douglas Conjecture

Arveon-Douglas Conjecture: I a homogenous ideal in $\mathbb{C}[z_1, \ldots, z_n]$, then $\forall p > \dim_{\mathbb{C}} Z(I)$, $[I]^{\perp}$ is p-essentially normal, i.e., $[S_i, S_j^*] \in \mathcal{S}^p$, where $S_i = S_{z_i}$.

Other forms: p- essentially normal, $p > n$; essentially normal ($[S_i, S_j^*] \in \mathcal{K}$).

Equivalent(when I homogenous): on Weighted Bergman spaces $L^2_{a,s}(\mathbb{B}_n)$, on the Hardy space $H^2(\mathbb{B}_n)$

Early results

I generated by monomials - Arveson

$n \leq 3$, homogenous - Guo, K.Wang

I principal in $L^2_a(\mathbb{B}_n)$- Douglas, K.Wang

in $H^2(\mathbb{B}_n)$ - Fang, Xia

Geometric Arveson-Douglas Conjecture

$\mathcal{H} = H_n^2, L_{a,s}^2, H^2$, M a homogenous variety in $\mathbb{B}_n$,

$$P_M = \{f \in \mathcal{H}, f|_M = 0\}$$

defines a submodule. $Q_M = P_M^{\perp}$ the quotient module.

Geometric Arveson-Douglas Conjecture:
$\forall p > \dim_{\mathbb{C}} M$, Q_M is p-essentially normal.

Early results

Kennedy and Shalit studied decomposition of varieties

Engliš and Eschmeier - M homogenous, smooth away from 0 in H_n^2

Douglas, Tang and Yu - M complete intersection, smooth on $\partial\mathbb{B}_n$, transverse with $\partial\mathbb{B}_n$ in $L_a^2(\mathbb{B}_n)$

Four Equivalence

Arveson showed that

Lemma

P is a submodule of $\mathcal{H}$, $Q = P^{\perp}$ is the quotient module, TFAE

(1) *P is essentially normal*

(2) *Q is essentially normal*

(3) *The commutators $[M_i, P]$ are compact*

(4) *The commutators $[M_i, Q]$ are compact.*

Main Result I

Theorem (Douglas, Y. Wang)

M is smooth on $\partial\mathbb{B}_n$, transverse with $\partial\mathbb{B}_n$, then the quotient module Q_M in $L_a^2(\mathbb{B}_n)$ is p-essentially normal for $p > 2\dim_{\mathbb{C}} M$.

Introduced complex harmonic analysis methods.

Showed connection with problem of holomorphic extension

Carleson Measure

Carleson Measure: A positive measure μ on $\mathbb{B}_n$ is a Carleson Measure if there is a constant $C > 0$ such that

$$\int |f|^2 d\mu \leq C\|f\|^2, \quad \forall f \in L_a^2.$$

Can define operator $T_\mu \in \mathcal{B}(L_a^2(\mathbb{B}_n))$

$$T_\mu f(z) = \int f(w) K_w(z) d\mu(w)$$

Complex Harmonic Analysis Method

We obtained the following: If there exists an "equivalent" Carleson measure μ supported on M, i.e., there exists $C > c > 0$,

$$c\|f\|^2 \leq \int_M |f|^2 d\mu \leq C\|f\|^2, \forall f \in Q_M$$

Then the quotient module Q_M is essentially normal.

Similarity of modules

Moreover, when this is true, the projection operator Q_M is in the Toeplitz algebra $\mathcal{T}(L^\infty)$.
In fact, $Q_M \in C^*(T_\mu)$.
And we have the following “almost” similarity of modules .

$$Q_M \simeq \mathcal{P}(\mu) \subseteq L^2_{a,s}(M)$$

$(mod \mathcal{S}^p), \forall p > 2 \dim_{\mathbb{C}} M$

Holomorphic Extension

Theorem (Beatrous)

M smooth, transverse with $\partial\mathbb{B}_n$, then there exists a bounded extension operator

$$E : L^2_{a,s}(M) \to L^2_a(\mathbb{B}_n),$$

$s = n - \dim M$. $Ef|_M = f$.

Connection with Holomorphic Extension

If such an extension operator exists, then the weighted measure is an “equivalent measure”.

On the other hand, any “equivalent measure” gives some kind of extension operator.

Decomposition of Varieties

Question: If M can be decomposed into "nice" varieties, $M = M_1 \cup M_2$, when is Q_M essentially normal?

When the projections Q_{M_i} are in Toeplitz algebra, can apply Suárez's localization technique

Theorem (Suárez)

$S \in \mathcal{T}(L^\infty)$, *then*

$$\|S\|_e = \sup_{x \in M_A \setminus \mathbb{B}_n} \|S_x\|.$$

Here S_x is a localization operator defined by Suárez.

We studied the localization operators of projections Q_{M_i} and used it to study their "essential angle".

Main Result II

Theorem

$M = M_1 \cup M_2$, $M_3 = M_1 \cap M_2$. If

(i) *M_1, M_2 and M_3 transverse with $\partial\mathbb{B}_n$, smooth on $\partial\mathbb{B}_n$.*

(ii) *M_1 and M_2 intersect cleanly on $\partial\mathbb{B}_n$.*

Then

$Q_1 \cap Q_2 / Q_3$ is finite dimensional and $Q_1 + Q_2$ is closed. As a consequence, $Q_1 + Q_2$ is p-essentially normal for $p > 2d$, where $d = \max\{\dim M_1, \dim M_2\}$.

Principal Submodules

Theorem (Douglas, K.Wang)

$I =< p >$ is principal, then $[I] \subseteq L^2_a(\mathbb{B}_n)$ is p essentially normal for all $p > n$.

Proof uses harmonic analysis but is very technical.

Simplified Proof

We found a simplified proof based on the following observation

$$
\begin{aligned}
& M_{z_i}^*(pf)(z) - p(z)M_{z_i}^*(f)(z) \\
= & \int (\bar{w}_i - \bar{z}_i)(p(w) - p(z))f(w)K_w(z)dv(w)
\end{aligned}
$$

Simplified Proof II

and a key estimation

$$|p(z)f(w)| \lesssim \frac{|1-\langle w,z\rangle|^d}{(1-|w|^2)^{d+n+1}} \int_{D(w,1)} |p(\lambda)f(\lambda)|dv(\lambda)$$

$\forall f \in L_a^2(\mathbb{B}_n)$, where $D(w,1)$ is the hyperbolic ball centered at w, with radius 1.

Main Result III

We extended the result to pseudoconvex domains.

Theorem

$\Omega \subseteq \mathbb{C}^n$ a bounded strongly pseudoconvex set with smooth boundary, $p \in \mathbb{C}[z_1, \ldots, z_n]$, then the submodule $[p] \subseteq L^2_a(\Omega)$ is p-essentially normal for all $p > n$.

Main Result VI

And for non-polynomial generator

Theorem

$h \in \mathcal{O}(\overline{\mathbb{B}_n})$, then the principal submodule generated by h, $[h] \subseteq L^2_a(\mathbb{B}_n)$ is p-essentially normal, $\forall p > n$.

Still progressing...

Thank you!