Classification of C*-envelopes of tensor algebras arising from stochastic matrices

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Joint Work with

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Multivariable Operator Theory at the Technion
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Dor-On-M.'16 Adam Dor-On and Daniel Markiewicz, "C*-envelopes of tensor algebras arising from stochastic matrices", *Integral Equations and Operator Theory* (2017), doi:10.1007/s00020-017-2382-x (also in the arXiv).

General Problem

What is the C*-envelope of the Tensor Algebra of the subproduct system over $\mathbb{N}$ arising from a stochastic matrix?

There are some surprises when compared to the situation of *product* systems over $\mathbb{N}$.

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Definition (Shalit-Solel '09, Bhat-Mukherjee '10)

Let M be a vN algebra, let $X = (X_n)_{n\in\mathbb{N}}$ be a family of W*-correspondences over M, and let $U = (U_{m,n} : X_m \otimes X_n \to X_{m+n})$ be a family of bounded M-linear maps. We say that X is a subproduct system over M if for all $m, n, p \in \mathbb{N}$,

1. $X_0 = M$
2. $U_{m,n}$ is co-isometric
3. The family U "behaves like multiplication": $U_{m,0}$ and $U_{0,n}$ are the right/left multiplications and

$$U_{m+n,p}(U_{m,n} \otimes I_p) = U_{m,n+p}(I_m \otimes U_{n,p})$$

When $U_{m,n}$ is unitary for all m, n we say that X is a product system.

Theorem (Muhly-Solel '02, Solel-Shalit '09)

Let M be a vN algebra. Suppose that $\theta : M \to M$ is a unital normal CP map. Then there exits a canonical subproduct system structure on the family of Arveson-Stinespring correspondences associated to $(\theta^n)_{n\in\mathbb{N}}$.

Definition

Given a countable (possibly infinite) set Ω, a stochastic matrix over Ω is a function $P : \Omega \times \Omega \to \mathbb{R}$ such that $P_{ij} \geq 0$ for all i, j and $\sum_{j\in\Omega} P_{ij} = 1$ for all i.

Subproduct system of a stochastic matrix

There is a 1-1 correspondence between ucp maps of $\ell^\infty(\Omega)$ into itself and stochastic matrices over Ω given by

$$\theta_P(f)(i) = \sum_{j\in\Omega} P_{ij} f(j)$$

Hence, a stochastic P gives rise to a canonical subproduct system $Arv(P)$.

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Given a subproduct system (X, U), we define the Fock W*-correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator

$$S_\xi^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n$$

- Tensor algebra (not self-adjoint):

$$\mathcal{T}_+(X) = \overline{\mathrm{Alg}}^{\|\cdot\|} M \cup \{S_\xi^{(m)} \mid \forall \xi \in X_m, \forall m\}$$

- Toeplitz algebra: $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$
- Cuntz-Pimsner algebra: $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$ for appropriate $\mathcal{J}(X)$

For the case of subproduct systems, Viselter '12 defined the ideal $\mathcal{J}(X)$ as follows: let Q_n denote the orthogonal projection onto the nth summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n\to\infty} \|TQ_n\| = 0\}.$$

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Example (Product system $\mathscr{P}^{\mathbb{C}}$)

Let $X = \mathscr{P}^{\mathbb{C}} = \cup_{n\in\mathbb{N}}\mathbb{C}$ be the "line bundle" product system.

- We have $\mathcal{F}_X = \oplus_{n\in\mathbb{N}}\mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathscr{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathscr{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathscr{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathscr{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system $\mathscr{P}_E$ is faithful, then $\mathcal{O}(\mathscr{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

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In a previous paper with A. Dor-On, we studied the tensor algebras in their own right. Let's do a quick review.

- Recall that a stochastic matrix P is essential if for every i, $P^n_{ij} > 0$ for some n implies that $\exists m$ such that $P^m_{ji} > 0$.
- The support of P is the matrix $\mathrm{supp}(P)$ given by

$$\mathrm{supp}(P)_{ij} = \begin{cases} 1, & P_{ij} \neq 0 \\ 0, & P_{ij} = 0 \end{cases}$$

Theorem (Dor-On-M.'14)

Let P and Q be finite stochastic matrices over Ω. TFAE:

1. *There is an algebraic isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.*
2. *there is a graded comp. bounded isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.*
3. *$\mathrm{Arv}(P)$ and $\mathrm{Arv}(Q)$ are similar up to change of base*

Furthermore, if P and Q are essential , those conditions hold if and only if P and Q have the same supports up to permutation of Ω.

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Furthermore, if P and Q are essential , those conditions hold if and only if P and Q have the same supports up to permutation of Ω.

- A stochastic matrix P is recurrent if $\sum_n (P^n)_{ii} = \infty$ for all i.

Theorem (Dor-On-M.'14)

Let P and Q be stochastic matrices over Ω. TFAE:

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2. *there is a graded comp. isometric isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.*
3. *$\mathrm{Arv}(P)$ and $\mathrm{Arv}(Q)$ are unitarily isomorphic up to change of base.*

Furthermore, if P and Q are recurrent, those conditions hold if and only if P and Q are the same up to permutation of Ω.

We also computed the Cuntz-Pimsner algebra in the sense of Viselter.

Theorem (Dor-On-M.'14)

If P is irreducible $d \times d$ stochastic, then $\mathcal{O}(P) \simeq C(\mathbb{T}) \otimes M_d(\mathbb{C})$.

We thank Dilian Yang for pointing out a gap, fixed in Dor-On-M.'16.

We will turn the uncomplicated nature of $\mathcal{O}(P)$ to our advantage to study the C*-envelope of $\mathcal{T}_+(P)$.

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Definition (C*-envelope - existence proved by Hamana '79)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra. The C*-envelope of $\mathcal{A}$ consists of a C*-algebra $C^*_{\text{env}}(\mathcal{A})$ and a comp. isometric embedding $\iota : \mathcal{A} \to C^*_{\text{env}}(\mathcal{A})$ with the following universal property: if $j : \mathcal{A} \to B$ is a comp. isometric embedding and $B = C^*(j(A))$, then there is a *-homomorphism $\phi : B \to C^*_{\text{env}}(\mathcal{A})$ such that $\phi(j(a)) = \iota(a)$ for all $a \in \mathcal{A}$.

Definition (Arveson '69)

Let $\mathcal{S}$ be an operator system. We say that a UCP map $\phi : \mathcal{S} \to B(H)$ has the unique extension property (UEP) if it has a unique cp extension $\tilde{\phi} : C^*(\mathcal{S}) \to B(H)$ which is a $*$-rep. If $\tilde{\phi}$ is irreducible, then ϕ is called a boundary representation of $\mathcal{S}$.

Theorem (Arveson '08 for $\mathcal{A}$ separable, Davidson-Kennedy '13)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra and let $S = \mathcal{A} + \mathcal{A}^*$. Let π be the direct sum of all boundary representations of $\mathcal{A}$. Then the C-envelope of $\mathcal{A}$ is given by the pair $\pi \restriction_{\mathcal{A}}$ and $C^*(\pi(S))$.*

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Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C-correspondence, then $C^*_{\mathrm{env}}(\mathcal{T}_+(E)) = \mathcal{O}(E)$.*

Theorem (Davidson, Ramsey and Shalit '11)

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- Recall if P is irreducible finite stochastic, $\mathcal{O}(P) \simeq C(\mathbb{T}) \otimes M_d(\mathbb{C})$.
- Let $H = \mathcal{F}_{Arv(P)} \otimes \ell^2(\Omega)$. We have a canonical representation $\pi : \mathcal{T}(P) \to B(H)$ which breaks up into d subrepresentations π_k on the "column-like" spaces $H_k = \mathcal{F}_{Arv(P)} \otimes \mathbb{C}e_k$.

Theorem (Dor-On-M.'16)

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Theorem (Dor-On-M.'16)

Suppose that P is an irreducible matrix of size d. The point evaluations of $C(\mathbb{T}) \otimes M_d(\mathbb{C})$ lift to boundary representations of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$. Therefore have an exact sequence

$$0 \longrightarrow \bigoplus_{j \in \Omega_b^P} \mathbb{K}(H_j) \longrightarrow C^*_{\mathrm{env}}(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_d \longrightarrow 0$$

where Ω_b^P is the set of states k for which π_k is boundary.

Definition

Let P be an irreducible r-periodic stochastic matrix of size d. A state $k \in \Omega$ is called exclusive if whenever for $i \in \Omega$ and $n \in \mathbb{N}$ we have $P_{ik}^{(n)} > 0$, then $P_{ik}^{(n)} = 1$.

We say that P has the multiple-arrival property if whenever $k, s \in \Omega$ are distinct non-exclusive states such that whenever k leads to s in n steps, then there exists $k \neq k' \in \Omega$ such that k' leads to s in n steps.

Example

If P is r-periodic, then by permuting states it has the cyclic block decomposition

$$\begin{bmatrix} 0 & P_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{r-2} \\ P_{r-1} & \cdots & 0 & 0 \end{bmatrix}, \qquad \text{example: } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

If such a matrix has full-support, which is to say no zeros in the blocks P_j, then it has multiple-arrival.

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Theorem (Dor-On-M.'16)

Let P be an irreducible finite stochastic matrix. If $k \in \Omega$ is exclusive, then π_k is not a boundary rep.

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Suppose that P is a finite irreducible matrix with multiple-arrival. Then π_k is a boundary representation if and only if k is non-exclusive. Therefore, the C-envelope of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$ corresponds to the quotient by the ideal*

$$\bigcap_{k \text{ non-exclusive}} \{ T \in \mathcal{J}(P) \mid \pi_k(T) = 0 \} \simeq_\pi \bigoplus_{j \text{ exclusive}} \mathbb{K}(H_j)$$

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Example (Dor-On-M.'16: Dichotomy fails)

$C^*_{\text{env}}(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ and $\mathcal{O}(P)$ are all different for $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$.

Since P is 2-periodic, we see from its cyclic decomposition it has full-support. Therefore it has the multiple-arrival property. The only exclusive column is $k = 3$. Therefore we have an exact sequence

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Recall $\Omega_b^P = \{k \in \Omega : \pi_k$ is boundary for $P\}$

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Let P be a finite irreducible stochastic.

1. *If P has a non-exclusive state then*

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Let P be an r-periodic irreducible stochastic matrix over Ω of size d, and $k \in \Omega$. Let $\Omega_0, ..., \Omega_{r-1}$ be a cyclic decomposition for P, so that $\sigma(k)$ is the unique index such that $k \in \Omega_{\sigma(k)}$. The k-th column nullity of P is

$$\mathcal{N}_P(k) = \sum_{m=1}^{\infty} |\{ \, i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} = 0 \, \}|$$

Intuition: It counts the number of zeros in the kth column of the powers of P, relative to the cyclic decomposition support.

$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \to \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \to \dots$$

Note the series is actually a sum, because the matrix powers fill-out eventually.

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*Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $C^*_{\text{env}}(\mathcal{T}_+(P))$ and $C^*_{\text{env}}(\mathcal{T}_+(Q))$ are *-isomorphic if and only if*

1. $|\Omega^P| = |\Omega^Q|$ *(let d be this number)*
2. *there is a bijection $\tau : \Omega^P_b \to \Omega^Q_b$ such that*

$$\forall k \in \Omega^P_b, \qquad \mathcal{N}_P(k) \equiv \mathcal{N}_Q(\tau(k)) \mod d.$$

Example

Suppose matrices for P, Q, R are stochastic with matrices supported on graphs (so multiple-arrival)

$$Gr(P) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \; Gr(Q) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \; Gr(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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$$\Omega_b^R = \{1,2,3\}, \qquad \mathcal{N}_R(1) = \mathcal{N}_R(2) = 0, \; \mathcal{N}_R(3) = 1,$$

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Thank you!

Extension theory:

$$0 \to K \xrightarrow{\iota} A \xrightarrow{\pi} B \to 0$$

can be studied through Busby invariant $\eta : B \to Q(K) \cong M(K)/K$, since have $\theta : A \to M(K)$ by $\theta(a)c = \iota^{-1}(a\iota(c))$

Equivalence of exact sequences gives relation for Busby inv.: $\exists \kappa : K_1 \to K_2$ and $\beta : B_1 \to B_2$ s.t. $\tilde{\kappa}\eta_1 = \eta_2\beta$.

In our case closely connected to $K = \mathbb{K}$ for which a lot is known. There is a group structure on the set of equivalence classes of extensions (both weak and strong) since B is nuclear separable (Choi-Effros).

$$\mathrm{Ext}_s(B) \to \mathrm{Ext}_w(B) \to \mathrm{Hom}(K_1(B), \mathbb{Z})$$

We use work of Paschke and Salinas, which characterizes this objects when K is a sum of compacts, and sweat to identify the objects and maps in this case.

Example

Suppose matrices for P, Q, R are stochastic with matrices supported on graphs (so multiple-arrival)

$$Gr(P) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \ Gr(Q) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ Gr(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Omega_b^P = \{1,2\}, \qquad \mathcal{N}_P(j) = 0, \quad j = 1,2,3$$

$$\Omega_b^Q = \{1,2,3\}, \qquad \mathcal{N}_Q(1) = \mathcal{N}_Q(2) = 0, \ \mathcal{N}_Q(3) = 1,$$

$$\Omega_b^R = \{1,2,3\}, \qquad \mathcal{N}_R(1) = \mathcal{N}_R(2) = 0, \ \mathcal{N}_R(3) = 1,$$

$$C^*_{\text{env}}(\mathcal{T}_+(P)) \not\sim C^*_{\text{env}}(\mathcal{T}_+(Q)) \cong C^*_{\text{env}}(\mathcal{T}_+(R))$$

$$\mathcal{O}_{Gr(P)} \cong \mathcal{O}_{Gr(Q)} \not\sim \mathcal{O}_{Gr(R)}$$

where $\cong$ stands for *-isomorphism and $\sim$ stands for stable isomorphism.