LATTICES AND SAILS

The proposed project deals with understanding a geometric object called a *sail* which one attaches to a lattice in $\mathbb{R}^n$. The aim is to experience valuable new research - some of the questions which I will suggest are open. A lattice Λ in $\mathbb{R}^n$ is a subset of the form $\Lambda = \{\sum_1^n k_i v_i : k_i \in \mathbb{Z}\}$ where the v_i's form a basis of $\mathbb{R}^n$. The sail is defined as follows:

$$C_\Lambda = \partial \operatorname{conv}\left(\Lambda \cap \{w \in \mathbb{R}^n : w_i > 0 \text{ for all } 1 \leq i \leq n\}\right).$$

In words, it is the boundary of the smallest convex set containing the lattice points all of whose coordinates are positive. If you picture the sail of a lattice in $\mathbb{R}^2$ then unless you started with a lattice which contains a point on one of the axis, its simply a piecewise linear curve in the positive quadrant of the plane with asymptotes the positive x and y axis. The picture in higher dimensions is similar but more complicated. In general the sail is built out of $n-1$-dimensional polytopes whose vertices belong to the lattice and which we refer to as the *faces* of the sail.

I chose sails as the object of study in this project because of several reasons. The first is that it is highly elementary and there are elementary questions which we do not know the answer to (maybe only because they have not been investigated). The second reason is that the sail encodes lots of information. Explaining what information it encodes is more tricky and actually not needed for the project. As an example, if the sail is periodic (in a suitable sense), then the coordinates of the v_i's are essentially algebraic numbers.

Questions that we will address: Do the lattice points on the sail span the lattice? Do the lattice points on every face of the sail span the lattice? Can one bound the index of the lattice spanned by the lattice points in a face of the sail in the original lattice? What kind of special sails are there? What is a good notion for a random sail? What is a good notion of a periodic sail? (and the list goes on).

The prerequisites are: Linear algebra (must), topology and geometry in $\mathbb{R}^n$ (must). Other mathematical background that will be beneficial but is not essential is, measure theory, groups, rings, Galois theory.