Scalar conservation laws on graphs

A scalar conservation law on channel $I := [0, 1]$ is given by an equation of the form

$$\rho_t + J_x = 0 \quad (x, t) \in I \times \mathbb{R}^+ \tag{1}$$

Here $\rho = \rho(x, t)$ represents the density of a cloud of particles at $x \in I$ at time t, and $J = J(x, t)$ is the corresponding flux. The zero flux condition $J(0, t) = J(1, t) = 0$ on the boundary guarantees that the total mass is conserved:

$$\int_I \rho(x, t)dx = \int_I \rho(x, 0)dx := M \text{ for any } t > 0.$$

In the particular case $J = -\rho_x$ we recover the classical heat equation which, as you should know, is a well posed linear equation under the stated condition, and can even be solved by Fourier expansion. Physically, it may represents the evolution of the particles in the cloud subjected to a diffusion. You should also know that $\lim_{t\to\infty} \rho(x, t) = M$, independent of details of the initial data (if you forgot, see chapter 5 of [1], or its Hebrew version).

A more general flux is

$$J = -\rho_x + \rho V_x$$

where $V = V(x, t)$ is a prescribed, differential function. Then, the scalar conservation law is still a linear equation, representing the evolution of the cloud under the combined forces of diffusion and drift along the force V_x. If the potential $V = V(x)$ is independent of time, it admits a steady solution $\bar{\rho}(x) = \frac{M}{\int_I e^{V(y)}dy} e^{V(x)}$, which agrees with the no-flux on the boundary condition as well. Moreover, it is not difficult to show that, as for the heat equation, $\bar{\rho}$ is the limit, as $t \to \infty$, of the solution for any initial data of the prescribed mass $M > 0$.

In some models the potential V is not prescribed, but obtained due to self-interaction between particles in the cloud. If $K = K(x, y)$ is the interaction kernel between two particles at $x, y \in I$ then the potential is

$$V(x, t) := \int_I K(x, y)\rho(y, t)dy$$

and the equation is not linear (but still admits an interesting structure). For a comprehensive discussion of this, and related models see [2].

In this project we will attempt to generalize the above setting to the case where the channel I is replaced by a metric graph, i.e a collection of nodes and edges. We will try to understand what is the proper generalization of the no-flux conditions at the nodes of the graph, and to characterize the steady states under some assumptions on the kernel K.

The expected research is, essentially, a theoretical one, but we may use some numerical calculations to get the feeling of it.

Pre-requisite: A basic course in PDE (for example, 104030 for Technion students).

References

1. Y. Pinchover and J. Rubinstein, *Introduction to Partial Differential Equations*, Cambridge University Press, 2005

2. G. Wolansky: *On steady distributions of self-interacting clusters under friction and fluctuations*, Arch. Rat. Mech. Anal. **119**, 355-391, 1992