

### 3 Basic eigenvalue properties

#### 3.1 Monotonicity properties

We use here the variational characterization from the previous chapter to prove some monotonicity results.

Theorem 3.1: let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with Lipschitz boundary.

Then, the Neumann eigenvalues of the Laplacian lie below their Dirichlet counterparts,  $\mu_k \leq \lambda_k \quad \forall k \geq 1$ .

Proof: Recall the min-max theorems

$$\mu_k = \inf_{\mathcal{U}} \sup_{f \in \mathcal{U} \setminus \{0\}} R(f)$$

$$\lambda_k = \inf_V \sup_{f \in V \setminus \{0\}} R(f)$$

where  $V$  ranges over all the  $k$ -dimensional subspaces of  $H_0^1(\Omega)$  and  $\mathcal{U}$  ranges over all the  $k$ -dimensional subspaces of  $H^1(\Omega)$ .  
Each  $V$  is also a valid  $\mathcal{U}$  as  $H_0^1(\Omega) \subset H^1(\Omega)$  and since the second infimum is done over a subset of the set for which the first infimum is done, we have  $\mu_k \leq \lambda_k$ .  $\square$

## Domain monotonicity for Dirichlet spectrum

Theorem 3.2: let  $\Omega$  and  $\tilde{\Omega}$  be bounded domains in  $\mathbb{R}^n$  and denote their corresponding Dirichlet eigenvalues by  $\lambda_k, \tilde{\lambda}_k$  respectively.

If  $\Omega \supset \tilde{\Omega}$  then  $\lambda_k \leq \tilde{\lambda}_k \quad \forall k \geq 1$ .

Proof: By the min-max theorem we have

$$\lambda_k = \inf_V \sup_{f \in V \setminus \{0\}} R(f), \quad \tilde{\lambda}_k = \inf_{\tilde{V}} \sup_{f \in \tilde{V} \setminus \{0\}} R(f)$$

where  $V$  ranges over  $k$ -dim subspaces of  $H_0^1(\Omega)$  and  $\tilde{V}$  ranges over  $k$ -dim subspaces of  $H_0^1(\tilde{\Omega})$ .

Observe that each  $\tilde{V}$  is also a valid  $V$  since  $H_0^1(\tilde{\Omega}) \subset H_0^1(\Omega)$ , which holds as any approximating  $C^\infty(\tilde{\Omega})$  function is also in  $C^\infty(\Omega)$  once extended by 0 in  $\Omega \setminus \tilde{\Omega}$ .

The corresponding subspace  $\tilde{V}$  is not exactly the same as  $V$ , but its members have the same value of Rayleigh quotient (as they were extended by 0).

Therefore  $\lambda_k \leq \tilde{\lambda}_k$ . □

## Restricted reverse monotonicity for Neumann spectrum

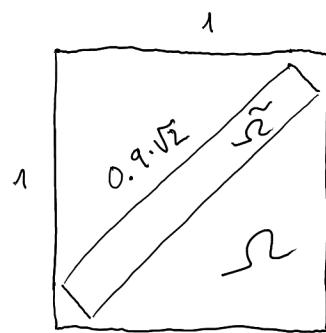
The proof above does not work for Neumann boundary, as  $H^1(\tilde{\Omega})$  is not a subspace of  $H^1(\Omega)$ . One may extend a  $H^1(\tilde{\Omega})$  function to form a function in  $H^1(\Omega)$ , but the extended function would not vanish in  $\Omega \setminus \tilde{\Omega}$  and its Rayleigh quotient will be generally different.

Monotonicity does hold under simple rescaling of the domain, as  $\mu_k(t\omega) = t^{-2} \mu_k(\omega) \leq \mu_k(\omega)$  for  $t \geq 1$ , so that larger domain does have smaller eigenvalues.

A (counter-) example for larger domain with larger eigenvalues:

$\omega$ : square of side=1

$\tilde{\omega}$ : rectangle of side=  $0.9\sqrt{2}$



$$\mu_2 = \pi^2, \quad \tilde{\mu}_2 = \frac{\pi^2}{(0.9\sqrt{2})^2}$$

$\uparrow$  does not depend on the smaller side of rectangle

### 3.2 Isoperimetric inequalities

Motivating question: which domain  $\Omega \subset \mathbb{R}^n$  maximizes\ minimizes a Dirichlet\ Neumann eigenvalue  $\lambda_k$ ?

Note that  $\lambda_k(x\Omega) = \frac{1}{x^2} \lambda_k(\Omega)$ .

Each eigenvalue  $\lambda_n$  can be arbitrarily small\ large.  
Hence, we need to fix scaling.

Choose to fix total volume of  $\Omega$ .

#### Example

① Take  $\Omega$  to be rectangle of volume 1,  
side lengths  $a, \frac{1}{a}$ . Consider Dirichlet boundary cond.

$$\lambda_1^{(0)}(\Omega) = \pi^2 \left( a^2 + \frac{1}{a^2} \right) \xrightarrow{a \rightarrow 0} \infty$$

Hence, no sense to maximization problem.



② Same problem, but with Neumann boundary condition.

For  $a$  large enough we have

$$\lambda_1^{(N)}(\Omega) = \frac{\pi^2}{a^2} \xrightarrow{a \rightarrow \infty} 0$$

Hence, no sense to minimization problem.

Before continuing with spectral isoperimetric inequalities, we deal with the (ancient) geometric isoperimetric inequality.

## Co-area formula

Let  $\Omega \subset \mathbb{R}^n$  and  $F: \Omega \rightarrow [a, b]$  Lipschitz and  $h \in L^1(\Omega)$ .

Then:  $\int_{\Omega} h(x) dx = \int_a^b dt \int_{F^{-1}(t)} \frac{h(x)}{|\nabla F(x)|} dH_{n-1}(x), \quad (3.1)$

where  $H_{n-1}$  is the  $n-1$  dim' Hausdorff measure on  $F^{-1}(t)$ .

## Special cases

- Choosing  $h=1$  gives  $\int_{\Omega} |\nabla F(x)| dx = \int_a^b dt \int_{F^{-1}(t)} dH_{n-1}(x)$
- Choosing  $F(x) = |x|$  gives  $\int_{\Omega} h(x) dx = \int_{\min h}^{\max h} dr \int_{\partial B_r} h ds$   
integration in spherical coordinates

Explanation of the co-area formula:

We actually change variables in integration:

$$x_1' = x_1, \quad x_2' = x_2, \dots, \quad x_{n-1}' = x_{n-1}, \quad x_n' = F(x_1, \dots, x_n)$$

volume element:  $dx_1' \dots dx_n' = \left| \frac{\partial F}{\partial x_n} \right| \cdot dx_1 \dots dx_n$

$$\begin{aligned} \int_{\Omega} h dx_1 \dots dx_n &= \int_{\Omega'} \frac{h}{\left| \frac{\partial F}{\partial x_n} \right|} \cdot dx_1' \dots dx_n' = \int_a^b dt \int_{x_n'=t} \frac{h}{\left| \frac{\partial F}{\partial x_n} \right|} \cdot dx_1' \dots dx_{n-1}' \\ &= \int_a^b dt \int_{x_n'=t} \frac{h}{|\nabla F|} \cdot \frac{|\nabla F|}{\left| \frac{\partial F}{\partial x_n} \right|} dx_1' \dots dx_{n-1}' = \int_a^b dt \int_{x_n'=t} \frac{h}{|\nabla F|} dH_{n-1} \end{aligned}$$

Assuming in the above  $\frac{\partial F}{\partial x_n} \neq 0$  and by implicit function theorem there exist a function  $g(x_1, \dots, x_{n-1})$  such that...

$$F(x_1, \dots, x_{n+1}, g(x_1, \dots, x_{n+1})) = t$$

↳ on this level set of  $F$  we have

$$0 = \frac{dF}{dx_i} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial x_n} \cdot \frac{\partial g}{\partial x_i}$$

$$\sqrt{1 + \left(\frac{\partial g}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial g}{\partial x_{n+1}}\right)^2} = \sqrt{1 + \left(\frac{F_{x_1}}{F_{x_n}}\right)^2 + \dots + \left(\frac{F_{x_{n+1}}}{F_{x_n}}\right)^2} = \frac{|\nabla F|}{|\frac{\partial F}{\partial x_n}|}$$

and the volume hyper-surface element on  $F^{-1}(t)$  is indeed

$$\sqrt{1 + \left(\frac{\partial g}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial g}{\partial x_{n+1}}\right)^2} dx_1^1 \cdots dx_{n+1}^1$$

(a normal to the hyper-surface element is  $\left(-\frac{\partial g}{\partial x_1}, \dots, -\frac{\partial g}{\partial x_n}, 1\right)$ )

### Spectral isoperimetric inequality - Dirichlet

Theorem 3.3 (Faber-Krahn inequality)

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and let  $B \subset \mathbb{R}^n$  be a ball with  $\text{vol}(\Omega) = \text{vol}(B)$ . Then

$$\lambda_1(\Omega) \geq \lambda_1(B),$$

where  $\lambda_1(\Omega)$ ,  $\lambda_1(B)$  are the first Dirichlet eigenvalues of  $\Omega$ ,  $B$ .

Remark: We saw (beginning of chapter) that maximization problem for Dirichlet has no sense.

Here we see that the minimizer is a ball.

Proof of thm 3.3: For  $F: \Omega \rightarrow \mathbb{R}$  we denote

$$\Omega_t := \{x \in \mathbb{R}^n \mid F(x) > t\}$$

Define a symmetrization  $F_*: \mathbb{B} \rightarrow [0, +\infty)$  of  $F$ :  
let  $B_t$  be a ball centered at the origin  
satisfying  $\text{vol}(B_t) = \text{vol}(\mathcal{S}_t)$ .

The symmetrization  $F_*$  is defined as the radially symmetric function such that

$$\{x \in \mathbb{R}^n \mid F_*(x) > t\} = B_t$$

By the co-area formula (with  $h=1$ )

$$\int_t^{\max F} ds \int_{F^{-1}(s)} \frac{1}{|\nabla F|} dH_{n-1}(x) = \int_{-t}^t 1 dx = \text{vol}(\mathcal{S}_t)$$

$$= \text{vol}(B_t) = \int_t^{\max F_*} ds \int_{F_*^{-1}(s)} \frac{1}{|\nabla F_*|} dH_{n-1}(x)$$

Differentiating with respect to  $t$ :

$$\int_{F^{-1}(t)} \frac{1}{|\nabla F|} dH_{n-1}(x) = \int_{F_*^{-1}(t)} \frac{1}{|\nabla F_*|} dH_{n-1}(x) \quad \#t \quad (\times)$$

Also,

$$\int_{\mathbb{B}} F^2 dx = \int_0^{\max F} ds \int_{F^{-1}(s)} \frac{F^2}{|\nabla F|} dH_{n-1}$$

$$= \int_0^{\max F} s^2 ds \int_{F^{-1}(s)} \frac{1}{|\nabla F|} dH_{n-1}$$

$$(*) \Rightarrow \int_0^{\max F} s^2 ds \int_{F_*^{-1}(s)} \frac{1}{|\nabla F_*|} dH_{n-1} = \int_{\mathbb{B}} F_*^2 dx \quad (\times)$$

The last equality follows by the same steps, but in the other direction and taking into account that  $\max F = \max F_*$ .

For  $t \in [0, \max F]$  define:

$$G(t) = \int_{\Omega_t} |\nabla F|^2 dx \quad \text{and} \quad G_*(t) = \int_{B_t} |\nabla F_*|^2 dx$$

By the co-area formula (with  $h = |\nabla F|^2$ )

$$G(t) = \int_t^{\max F} ds \int_{F(s)} |\nabla F| dH_{n-1}$$

$$\Rightarrow G'(t) = - \int_{F^{-1}(t)} |\nabla F| dH_{n-1}$$

and similarly:  $G'_*(t) = - \int_{F_*^{-1}(t)} |\nabla F_*| dH_{n-1}$

From Cauchy-Schwartz inequality:

$$(\text{vol}(F^{-1}(t)))^2 = \left( \int_{F^{-1}(t)} 1 ds \right)^2 \leq \left( \int_{F^{-1}(t)} |\nabla F| dH_{n-1} \right) \cdot \left( \int_{F^{-1}(t)} \frac{1}{|\nabla F|} dH_{n-1} \right)$$

using that  $|\nabla F_*|$  is constant on  $F_*^{-1}(t)$  (being radially symmetric):

$$(\text{vol}(F_*^{-1}(t)))^2 = \left( \int_{F_*^{-1}(t)} 1 ds \right)^2 = \left( \int_{F_*^{-1}(t)} |\nabla F_*| dH_{n-1} \right) \left( \int_{F_*^{-1}(t)} \frac{1}{|\nabla F_*|} dH_{n-1} \right)$$

By the geometric isoperimetric inequality:

$$\text{vol}(\Omega_t) = \text{vol}(B_t) \Rightarrow \text{vol}(F^{-1}(t)) \geq \text{vol}(F_*^{-1}(t))$$

Combining the three last identities with (\*) we get

$$\forall t \quad \int_{F^{-1}(t)} |\nabla F| dH_{n-1} \geq \int_{F_*^{-1}(t)} |\nabla F_*| dH_{n-1}$$

We get that:

$$G'(t) = - \int_{F^1(t)} |\nabla F| dH_m \leq - \int_{F_*^1(t)} |\nabla F_*| dH_{m-1} = G'_*(t)$$

We integrate this with respect to  $t$

(using that  $G(\max F) = 0 = G_*(\max F_*)$ ) :

$$G(0) \geq G_*(0)$$

$$G(0) = \int_0^{\max F} ds \int_{F^1(s)} |\nabla F| dH_{m-1} = \int_{\Omega} |\nabla F|^2 dx$$

$\uparrow$   
co-area

and similarly  $G_*(0) = \int_B |\nabla F_*|^2 dx$

so that  $\int_{\Omega} |\nabla F|^2 dx \geq \int_B |\nabla F_*|^2 dx \quad (***)$

Now, we choose  $F$  to be the first eigenfunction of  $\Omega$ , corresponding to  $\lambda_1$  and get:

$$\lambda_1(\Omega) = \frac{\int_{\Omega} |\nabla F|^2 dx}{\int_{\Omega} |F|^2 dx} \geq \frac{\int_B |\nabla F_*|^2}{\int_B |F_*|^2} = R(F_*) \geq \lambda_1(B)$$

$\uparrow$   
 $(**), (***)$

Rayleigh quotient of a test function  $F_*$  is larger than the minimum.

□

Remarks:

- ① The symmetrization technique  $F \rightarrow F_*$  is also called Schwartz rearrangement (or spherical decreasing rearrangement).

② The ball is not the unique minimizer, as we can also remove from it finite number of points for example.

It is a unique minimizer, if such irregularities are not allowed.

### Spectral isoperimetric inequality - Neumann

Theorem 3.4: (Szegö-Weinberger)

The ball maximizes the second Neumann eigenvalue (i.e, first positive eigenvalue) among Lipschitz open sets of given volume.

Moreover, it is the only maximizer in this class.

Proof uses rearrangement and further methods.  
(not specified here).

3.3

## Cheeger's inequality

An inequality giving a geometric (Cheeger constant) bound on  $\lambda_1$ .

Note that using test functions in Rayleigh quotient it is easy to get upper bounds on  $\lambda_1$ .

Cheeger's inequality gives lower bound on  $\lambda_1$ .

Def 3.5:

① Let  $\Omega \subset \mathbb{R}^n$  bounded domain.

The Cheeger constant of  $\Omega$  is

$$h^{(D)}(\Omega) := \inf_{S \subseteq \Omega} \frac{\text{vol}_{n-1}(2S)}{\text{vol}_n(S)} \quad (3.2)$$

② Let  $\Omega$  be an  $n$ -dim smooth, compact, Riemannian manifold w/o boundary.

The Cheeger constant of  $M$  is:

$$h^{(N)}(\Omega) := \inf_{S \subseteq M} \frac{\text{vol}_{n-1}(2S)}{\min(\text{vol}_n(S), \text{vol}_n(M \setminus S))} \quad (3.3)$$

Theorem 3.6:

① For  $\Omega$  with boundary  $\lambda_1^{(D)}(\Omega) \geq \frac{1}{4} h^{(D)}(\Omega)^2$   
Dirichlet eigenvalue

② For  $\Omega$  w/o boundary  $\lambda_1^{(N)}(\Omega) \geq \frac{1}{4} h^{(N)}(\Omega)$

Remarks ①  $M$  has no boundary, so  $\lambda_1(M) = 0$ .

② Cheeger constant has dimensions (units)

③ The ratio we minimize over in cheeger's constant is smaller whenever  $S$  is closer to a ball.  
(from geometric isoperimetric inequality).

Proof ①

Let  $u \in C_0^\infty(\Omega)$  such that  $u \geq 0$ .

$$\int_{\Omega} |\nabla u| dx = \underset{\substack{\text{co-area} \\ \text{formula}}}{\int_0^\infty} dt \int_{u=t} \frac{|\nabla u|}{|\nabla u|} dH_{n-1}(t) = \int_0^\infty dt \cdot \text{Vol}_{n-1}(\{u=t\})$$
$$\stackrel{(3.2)}{\geq} h \cdot \int_0^\infty dt \text{Vol}_n(\{u \geq t\}) = h \cdot \int_{\Omega} u dx \quad (\star)$$

Let  $\varphi$  be the eigenfunction corresponding to  $\lambda_1^{(0)}(\Omega)$ .  
Apply the above for  $u = \varphi^2$ :

$$\int_{\Omega} \nabla(\varphi^2) dx \geq h \int_{\Omega} \varphi^2 dx$$

Manipulate the L.H.S:

$$\int_{\Omega} \nabla(\varphi^2) dx = \int_{\Omega} 2\varphi \cdot \nabla \varphi dx \leq 2 \left( \int_{\Omega} \varphi^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \varphi|^2 dx \right)^{\frac{1}{2}} \quad (\star\star)$$

Cauchy  
Schwartz

$$\Rightarrow \lambda_1^{(0)}(\Omega) = \frac{\int_{\Omega} |\nabla \varphi|^2 dx}{\int_{\Omega} \varphi^2 dx} \geq \frac{h^2}{4}$$

② Take  $f$  to be a smooth function on  $\Omega$  s.t.

$$\int_{\Omega} f^2 dx = 1 \quad \int_{\Omega} f dx = 0 \quad \begin{bmatrix} \text{These are all admissible} \\ \text{functions in} \\ \lambda_1^{(m)} = \inf_f R(f) \end{bmatrix}$$

for any  $c \in \mathbb{R}$  we have

$$\int_{\Omega} (f+c)^2 dx = \int_{\Omega} f^2 dx + 2c \int_{\Omega} f dx + c^2 \text{vol}(\Omega) \geq \int_{\Omega} f^2 dx \quad (\star\star\star)$$

Take  $F := f+c$  and

$$S_1 := \{x \in \Omega \mid F(x) > 0\}, \quad S_2 := \{x \in \Omega \mid F(x) < 0\}$$

We may choose  $c$  s.t. both  $S_1, S_2$  have volume  $\leq \frac{1}{2} \text{vol}(\Omega)$ .

The Rayleigh quotient of the admissible function  $f$  is:

$$R(f) = \frac{\int_{\Omega} |\nabla f|^2}{\int_{\Omega} f^2} \stackrel{(\star\star\star)}{\geq} \frac{\int_{\Omega} |\nabla F|^2}{\int_{\Omega} F^2} = \frac{S_1 \int_{S_1} |\nabla F|^2 + S_2 \int_{S_2} |\nabla F|^2}{S_1 \int_{S_1} F^2 + S_2 \int_{S_2} F^2}$$

The proof will be over once we show

$$\int_{S_i} |\nabla F|^2 \geq \frac{1}{4} k^2(\Omega) \int_{S_i} F^2 \quad \text{for both } i=1,2.$$

This can be done just as in the proof of ①:  
choosing  $u = F^2$  and combining ① & ②.  $\square$

3.4

## Weyl's asymptotics

We denote  $\alpha_j \sim \beta_j$  for  $\lim_{j \rightarrow \infty} \frac{\alpha_j}{\beta_j} = 1$

$w_d :=$  volume of the unit ball in  $\mathbb{R}^d$ .

Theorem 3.7 (Weyl's law):

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$  with piecewise smooth boundary.

For both Dirichlet and Neumann we have:  $\lambda_j(\Omega) \sim 4\pi^2 \left( \frac{j}{w_d \cdot \text{vol}_d(\Omega)} \right)^{2/d}$  ( $d \geq 1$ )

In particular, for  $d=2$  we have  $w_2 = \pi$ :

$$\lambda_j(\Omega) \sim \frac{4\pi}{\text{Area}(\Omega)} \cdot j$$