Combinatorial games

Mentor: Ron Holzman

Combinatorial games are two-person games with perfect information and without chance, whose rules are naturally formulated in combinatorial terms. Classical examples include Tic-tac-toe, Connect 4, Nim, and Chomp. In every such game, one of the players has a winning strategy, or (if there is a possibility of a draw) both players have strategies guaranteeing them at least a draw. But determining which of these is the case for a specific game, or all games in a certain class, is a challenging task – undertaken by combinatorial game theory.

Consider the class of *superset games*. The game $S_{n,k}$, with integer parameters $0 \leq k \leq n$, is played on the collection of all sets $A \subseteq \{1, \ldots, n\}$ such that $1 \leq |A| \leq k$. Each player, at his turn, chooses a set in the collection which has not yet been removed, and removes it along with all its supersets. E.g., choosing $A = \{1, 3\}$ entails the removal of all sets B such that $\{1, 3\} \subseteq B$. The player whose turn it is to move when no sets are left, loses and his opponent wins.

Conjecture (Gale-Neyman, 1982): *The second player has a winning strategy in $S_{n,k}$ if and only if $k+1$ divides n.*

This 35 year old problem is still open. The conjecture is known to hold for $k \leq 2$ and for $k = n$, and also for certain pairs of small numbers (n, k).

The goal of this project is to understand the methods and ideas used to solve the known cases, and hopefully to resolve some new ones (e.g., the winner of $S_{6,5}$ and $S_{7,3}$ is not known). Variants of this class of games may also be studied.

No prior acquaintance with game theory is required, but a knack for combinatorial thinking and riddle solving is desirable.