INDEX OF FREE BOUNDARY MINIMAL SURFACES

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Introduction

Minimal surfaces are fascinating geometric objects that have been studied by many mathematicians starting from Lagrange in 1762. Loosely speaking, a surface Σ is minimal if it *locally* minimizes the area: if $\tilde{\Sigma}$ is obtained from Σ by perturbing it a little bit on a small enough set, then the area of $\tilde{\Sigma}$ is greater or equal to the area of Σ. However, the word "locally" is key here: the area does not have to increase under a deformation taking place on all the surface. Actually, the condition that the surface globally minimizes the area is quite restrictive: for example, it can be shown that the only minimal surface (without boundary or holes) in $\mathbb{R}^3$ that minimizes *globally* the area is the plane; yet there are many examples of minimal surfaces in $\mathbb{R}^3$ apart from the plane (see http://www.indiana.edu/~minimal/archive/index.html for beautiful pictures of some of them).

One can measure the defect of a minimal surface of having globally minimizing area by an integer called the *index*. Loosely speaking, the index measures how many global deformations of Σ decrease the area. Technically, for minimal surfaces of $\mathbb{R}^3$, the index is computed as the number of negative eigenvalues of the differential operator $-\Delta - 2K$ on Σ, where K is the Gauss curvature. A beautiful result that was shown in 1990 is the following characterization of low index minimal surfaces in the 3-dimensional sphere $\mathbb{S}^3$: the only minimal surfaces in $\mathbb{S}^3$ having index ≤ 5 are 2-dimensional spheres that are "equators" of $\mathbb{S}^3$ (index 1) and some tori that have index 5.

In the last few years, there has been a growing interest for *free boundary* minimal surfaces in the unit Euclidean ball; these are minimal surfaces with boundary inside the unit ball, whose boundary is contained in the unit sphere, and such that the surface intersects the unit sphere orthogonally. It turns out that an index can be defined for these surfaces.

The project

The goal of the project is to compute the index of a particular example of free boundary minimal surface. In the case we succeed in computing the index, this will be an interesting new result, however, most probably, the complete solution to this problem will take more than one week to achieve. This might even lead to a classification of low index free boundary surfaces similar to the one of the sphere. I expect that the proof will be mainly analytic, using only elementary spectral theory, namely solving some systems of ODE's –even though solving the ODE's in question might be quite tricky. During the first part of the project, the students will get familiar with the general theory of free boundary minimal surfaces, as well as with the particular example under investigation, and will try to make the connection between the index and the solutions to some systems of ODE's. Then, the rest of the week will be devoted to trying to solve these ODE's and use the solutions to compute the index. Also, even numerical solutions of these systems of ODE's, leading to some conjectural value of the index would be of great interest, and this might be already a very good goal, if some students in the group are motivated by numerical computations.

Students interested by this project are strongly encouraged to also attend the week of courses that will take place 3-8 September at the Technion, where a short introduction to spectral geometry will be given by R. Band.

Requirements

- A basic course in differential geometry covering regular surfaces, Gauss and mean curvature, second fundamental form, covariant derivative, Gauss-Bonnet formula.
- A course in ODE's.
- Some knowledge of Fourier series.