Representations of Toeplitz-Cuntz-Krieger algebras

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Motivation

- Perhaps the simplest example of a C^*-algebra with uncountably many unitarily inequivalent irreducible representations is the Cuntz algebra $\mathcal{O}_n$.
- $\mathcal{O}_n$ and $\mathcal{T}_n$ are the universal C^*-algebras generated by n isometries $S_i : \mathcal{H} \to \mathcal{H}$ such that $\sum_{i=1}^n S_i S_i^* = I_{\mathcal{H}}$ and $\sum_{i=1}^n S_i S_i^* \leq I_{\mathcal{H}}$ respectively.
- Glimm showed that one cannot classify all irreducible representations of $\mathcal{O}_n$ with countable structures. Instead, one looks at subclasses of those, or weakens the invariant.
- Irreducible representations of C^*-algebras are used in their classification, but results on them have applications to wavelets, fractals, and dynamical systems. *(!)*

Toeplitz-Cuntz-Krieger families

Let $G = (V, E, r, s)$ be a countable directed graph. A family $S = (S_v, S_e)_{v \in V, e \in E}$ of operators on a Hilbert space $\mathcal{H}$ is called a *Toeplitz-Cuntz-Krieger* family if

(P) $(S_v)_{v \in V}$ are pairwise orthogonal projections,

(IS) $S_e^* S_e = S_{s(e)}$ for all $e \in E$,

(TCK) $\sum_{e \in r^{-1}(v)} S_e S_e^* \leq S_v$ for all $v \in V$.

We say S is *Cuntz-Krieger* if additionally

(CK) $\sum_{e \in r^{-1}(v)} S_e S_e^* = S_v$ for all $v \in V$ with $0 < |r^{-1}(v)| < \infty$,

and *fully-coisometric* if additionally

(FC) $\sum_{e \in r^{-1}(v)} S_e S_e^* = S_v$ for all $v \in V$.

$\mathcal{T}(G)$ and $\mathcal{O}(G)$ are the universal C^* algebras generated by TCK and CK families respectively.

Left-regular representations

For a graph G we let $\mathbb{F}^+(G)$ be the set of all finite paths $\lambda = e_1 ... e_n$ in G where $s(e_i) = r(e_{i+1})$.

Example (Left-regular)

Let $\mathcal{H}_G = \ell^2(\mathbb{F}^+(G))$ *be the Hilbert space with o.n.b.* $\{\xi_\lambda\}_{\lambda \in \mathbb{F}^+(G)}$. *For* $v \in V$ *and* $e \in E$ *we have*

$$L_v(\xi_\lambda) = \begin{cases} \xi_\lambda & : \ r(\lambda) = v \\ 0 & : \ else \end{cases} \quad , \quad L_e(\xi_\lambda) = \begin{cases} \xi_{e\lambda} & : \ r(\lambda) = s(e) \\ 0 & : \ else \end{cases}$$

Then $L = (L_v, L_e)$ *is a TCK family, and we call the algebra*

$$\mathcal{L}_G := \overline{\mathrm{Alg}}^{\mathrm{WOT}} \{ \ L_\lambda \mid \lambda \in \mathbb{F}^+(G) \ \}$$

is called the left-regular free semigroupoid algebra.

History

- Row-contractions of operators were investigated in a series of papers by Popescu, generalizing many important results in dilation theory to the multivariable context. In fact, Popescu and Arias establish reflexivity and a Beurling type theorem for $\mathcal{L}_n$; the case of a single-vertex with n loops.
- Davidson and Pitts also establish these results at around the same time. They show hyperreflexivity of $\mathcal{L}_n$ and classify atomic representations up to unitary equivalence.
- Kribs–Power, Jury–Kribs and Katsoulis–Kribs generalize and expand many of the known results to arbitrary graphs. Among other things, they characterize semisimplicity and describe invariant subspaces of $\mathcal{L}_G$.

Wold-decomposition

We denote $\mathcal{H}_{G,w} := \overline{Sp}\{\xi_\lambda\}_{s(\lambda)=w}$, which is reducing for $L = (L_v, L_e)$, and denote $L_{G,w} := (L_v|_{\mathcal{W}_{G,w}}, L_e|_{\mathcal{W}_{G,w}})$.

Theorem (Wold-decomposition)

Let $S = (S_v, S_e)$ be a non-deg. TCK family for G. Then it is unitarily equivalent to $T \oplus \bigoplus_{v \in V} L_{G,v}^{(\alpha_v)}$, where T is a non-degenerate fully-coisometric family.

Definition

Let $S = (S_v, S_e)$ be a non-deg. TCK family. We call $\mathfrak{S} := \overline{\mathrm{Alg}}^{\mathrm{WOT}}\{S_\lambda\}_{\lambda \in \mathbb{F}^+(G)}$ a free semigroupoid algebra.

Clearly, if S and S' are unitarily equivalent, then $\mathfrak{S}$ and $\mathfrak{S}'$ are weak*-homeomorphic and completely isometrically isomorphic.

Wandering vectors

Definition

Let $\mathfrak{S}$ be a free semigroupoid algebra generated by a non-degenerate TCK family $S = (S_v, S_e)$ on $\mathcal{H}$.

1. *A vector $\xi \in \mathcal{H}$ is called wandering if $\{S_\lambda \xi\}_{\lambda \in \mathbb{F}^+(G)}$ is an orthogonal set.*
2. *We say that $\mathfrak{S}$ is analytic / type L if $\mathfrak{S}$ is weak* homeo. and completely isometrically isomorphic to $\mathcal{L}_{G_S}$ where G_S is the subgraph on v such that $S_v \neq 0$.*

Every free semigroupoid algebra on a space spanned by wandering vectors is analytic. In the single vertex case, we call $\mathfrak{S}$ a free semigroup algebra. Davidson, Katsoulis and Pitts prove a spatial structure theorem for them, and conjectured that every analytic free semigroup is spanned by wandering vectors. Kennedy was able to prove this conjecture is true.

Inductive type representation

Example (Inductive type)

Let $x = e_1 e_2 \ldots$ be an infinite backward path in G with $r(x) = v$. Define $x_m = e_1 \ldots e_m$ and let $\Gamma_x := \mathbb{F}^+(G)x^{-1}$ be elements of the form $\mu = \lambda x_m^{-1}$ in the free groupoid $\mathbb{F}(G)$ where we identify ee^{-1} with $r(e)$, and e identified with $r(e)e$ and $es(e)$. Take $\mathcal{H}_x := \ell^2(\Gamma_x)$ with o.n.b. $\{\xi_\mu\}_{\mu \in \Gamma}$. For $v \in V$ and $e \in E$ define

$$S_v(\xi_\mu) = \begin{cases} \xi_\mu & : \ r(\mu) = v \\ 0 & : \ else \end{cases} \quad , \quad S_e(\xi_\mu) = \begin{cases} \xi_\mu & : \ r(\mu) = s(e) \\ 0 & : \ else \end{cases}$$

Then $S = (S_v, S_e)$ is a fully-coisometric, and is spanned by wandering vectors. Hence $\mathfrak{S}$ is analytic, but is not left-regular.

Structure theorem

Theorem (Davidson, D., Li)

Let $\mathfrak{S}$ be a free semigroupoid algebra generated by $S = (S_v, S_e)$ on $\mathcal{H}$, of a graph G. Let $\mathfrak{M} = W^*(S)$ be the von-Neumann algebra generated by S. There is a projection $P \in \mathfrak{S}$ such that

1. *With respect to $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$ we have*

$$\mathfrak{S} = \begin{bmatrix} P\mathfrak{M}P & 0 \\ P^{\perp}\mathfrak{M}P & \mathfrak{S}P^{\perp} \end{bmatrix}$$

2. *If $\mathfrak{S} \neq \mathfrak{M}$ then $\mathfrak{S}P^{\perp}$ is analytic, isomorphic to $\mathcal{L}_{G'}$ where G' is the subgraph on vertices v such that $v \notin \overline{\langle S_e \rangle_{e \in E}}^{\mathrm{WOT}}$.*
3. *$P^{\perp}\mathcal{H}$ is spanned by wandering vectors. (!)*
4. *If each vertex is on a cycle, then P is the largest projection such that $P\mathfrak{S}P$ is self-adjoint. (!)*

Self-adjoint examples

A free semigroup algebra can be self-adjoint as Read was able to show. He produced a free semigroup algebra equal to $B(\mathcal{H})$.

Definition

Let G be a finite, transitive and d-in-degree regular graph.

1. *A strong edge coloring is a function $c : E \to \{1, 2, ..., d\}$ where $c(e) \neq c(f)$ for all $e \neq f$ in $r^{-1}(v)$ for $v \in V$.*
2. *A word $\gamma \in \mathbb{F}_d^+$ is called synchronizing for $v \in V$ if for any vertex $w \in V$ there's $\mu \in \mathbb{F}^+(G)$ from v to w with $c(\mu) = \gamma$.*

A famous conjecture of Adler and Weiss in graph theory is that G above is *aperiodic* iff some / all vertices have synchronizing words. It took 37 years until it was finally proven by Trahtman.

Theorem (Davidson, D., Li)

Suppose G is a finite, aperiodic, transitive and in-degree regular graph. Then there exists a CK family S such that $\mathfrak{S} = B(\mathcal{H})$.

History

- Absolute continuity of representations of $\mathcal{T}_n$ where $n \geq 2$ were introduced by Davidson, Li and Pitts in an attempt to better understand analytic free semigroup algebras.
- Kennedy showed that every absolutely continuous representation is analytic, and this was used to get an analogue of the Lebesgue-von-Neumann-Wold decomposition for isometries.
- Muhly and Solel investigated absolute continuity of representations of W^*-correspondences. They asked how far Kennedy's results on wandering vectors and absolute continuity can be stretched.

Dilations

If we have a family $A = (A_v, A_e)$ that satisfies the conditions of a TCK family, except that $S_e^* S_e \leq S_{s(e)}$ instead of (IS), we call A a contractive G-family.

Theorem (Bunce-Frahzo-Popescu; Muhly-Solel for C^*-cor.)

Let $A = (A_v, A_e)$ be a contractive G-family on $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a TCK family $S = (S_v, S_e)$ on $\mathcal{K}$ such that $P_{\mathcal{H}} S_\lambda|_{\mathcal{H}} = A_\lambda$ for every $\lambda \in \mathbb{F}^+(G)$, and S is the unique minimal dilation in the sense that the smallest S-invariant subspace of $\mathcal{K}$ containing $\mathcal{H}$ is $\mathcal{K}$, and any two such minimal TCK dilations are unitarily equivalent.

Contractive G-families are easy to produce, even in finite dimensional spaces. So an easy way to get examples of TCK families is to minimally dilate a contractive G-family.

Absolute continuity

Let $\mathcal{T}_+(G) = \overline{\mathrm{Alg}}^{\|\cdot\|}\{L_\lambda\}_{\lambda \in \mathbb{F}^+(G)}$ as a subalgebra of $\mathcal{T}(G)$.

Definition

Let $S = (S_v, S_e)$ be a TCK family on $\mathcal{H}$ for a graph G. S is

1. *absolutely continuous if for all $x, y \in \mathcal{H}$ there are $\xi, \eta \in \mathcal{H}_G$ such that $\langle \pi_S(A)x, y\rangle = \langle \pi_L(A)\xi, \eta\rangle$ for all $A \in \mathcal{T}_+(G)$,*
2. *singular if $\mathfrak{S}$ is a von-Neumann algebra,*
3. *of dilation type if S is the minimal dilation of the contractive G family $A = (PS_vP, PS_eP)$ on $P\mathcal{H}$.*

Theorem (Davidson, D., Li)

Let S be a is a TCK family of a non-cycle transitive graph. Then S is analytic if and only if it is absolutely continuous.

Lebesuge-von-Neumann-Wold decomposition

The following extends Kennedy's decomposition theorem to families of operators associated to some directed graphs.

Theorem (Lebesgue-von-Neumann-Wold decomposition; DDL)

Let S be a TCK family of a non-cycle transitive graph. Then up to unitary equivalence we may decompose,

$$S \cong S_\ell \oplus S_a \oplus S_s \oplus S_d$$

where

1. *S_ℓ is a left-regular TCK family.*
2. *S_a is an absolutely continuous fully-coisometric family.*
3. *S_s is a singular fully-coisometric family.*
4. *S_d is a dilation type fully-coisometric family.*

Consequences

As a consequence of a theorem of Katsoulis and Kribs and our structure theorem, we obtain an isomorphism theorem

Theorem (Davidson, D., Li)

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be nonselfadjoint free semigroupoid algebras for a transitive row-finite graphs G_1 and G_2 respectively. Then $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are algebraically isomorphic if and only if G_1 and G_2 are isomorphic graphs.

As a consequence of our absolute continuity results and methods of Davidson, Li and Pitts, we get a Kaplansky density theorem

Theorem (Davidson, D., Li)

Let S be a TCK family of a transitive non-cycle graph G. Then the unit of $\pi_S(\mathcal{T}_+(G))$ is weak dense in the unit ball of $\mathfrak{S}$.*

Ending

Thank you for your attention,

and Happy 65th birthday to Baruch Solel !