Hardy algebras associated with W^*-correspondences

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Introduction

We study tensor operator algebras (to be defined shortly) and their ultraweak closures: the **Hardy algebras**. I hope to present enough evidence for the claim that these algebras can be viewed as noncommutative $H^\infty(\mathbb{D})$.

To do this, we will consider these algebras as **algebras of (operator valued) functions** defined on the representation space of the algebra.

More precisely, we are led to consider a **family of functions** defined on a **family of sets**.

We shall discuss the **"matricial structure"** of this family of functions and their **"power series" expansions**.

♣ We were inspired by works of J. Taylor, D. Voiculescu, Kaliuzhnyi-Verbovetskyi and Vinnikov and Helton-Klepp-McCullough ,G. Popescu, K. Davidson and D. Pitts and others.

The Setup

$H^\infty(\mathbb{D})$ can be viewed as the algebra generated by I and the shift on ℓ^2. Our Hardy algebras are generated by a copy of a W^*-algebra and a collection of shifts.

More precisely, the set up is

$\diamond$ M - a W^*-algebra.

$\diamond$ E - a W^*-correspondence over M. This means that E is a **bimodule** over M which is endowed with an M-**valued inner product** (making it a right-Hilbert C^*-module that is self dual). The left action of M on E is given by a unital, normal, *-homomorphism φ of M into the $(W^*$-$)$ algebra of all bounded adjointable operators $\mathcal{L}(E)$ on E.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $d \geq 1$.

- $G = (G^0, G^1, r, s)$- a finite directed graph. $M = \ell^\infty(G^0)$,
  $E = \ell^\infty(G^1)$, $a\xi b(e) = a(r(e))\xi(e)b(s(e))$, $a, b \in M$,$\xi \in E$
  $\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$, $\xi, \eta \in E$.

- M- arbitrary , $\alpha : M \to M$ a normal unital, endomorphism.
  $E = M$ with right action by multiplication, left action by
  $\varphi = \alpha$ and inner product $\langle \xi, \eta \rangle := \xi^*\eta$. Denote it $_\alpha M$.

- Φ is a normal, contractive, CP map on M. $E = M \otimes_\Phi M$ is
  the completion of $M \otimes M$ with $\langle a \otimes b, c \otimes d \rangle = b^*\Phi(a^*c)d$
  and $c(a \otimes b)d = ca \otimes bd$.

Note: If σ is a representation of M on H, $E \otimes_\sigma H$ is a Hilbert
space with $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle_E)h_2 \rangle_H$.

Given two correspondences E and F over M, we can form the (internal) tensor product $E \otimes F$ by setting

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle f_1, \varphi(\langle e_1, e_2 \rangle_E) f_2 \rangle_F$$

$$\varphi_{E \otimes F}(a)(e \otimes f)b = \varphi_E(a)e \otimes fb$$

and applying an appropriate completion.
In particular we get "tensor powers" $E^{\otimes k}$.

Also, given a sequence $\{E_k\}$ of correspondences over M, the direct sum $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ is also a correspondence (after an appropriate completion).

For a correspondence E over M we define the Fock correspondence

$$\mathcal{F}(E) := M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$$

For every $a \in M$ define the operator $\varphi_\infty(a)$ on $\mathcal{F}(E)$ by

$$\varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

and $\varphi_\infty(a)b = ab$.
For $\xi \in E$, define the "shift" (or "creation") operator T_ξ by

$$T_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n.$$

and $T_\xi b = \xi b$. So that T_ξ maps $E^{\otimes k}$ into $E^{\otimes (k+1)}$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{T_\xi : \xi \in E\}$ will be called the **tensor algebra** of E and denoted $\mathcal{T}_+(E)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E)$ will be called the **Hardy algebra** of E and denoted $H^\infty(E)$.

Examples

1. If $M = E = \mathbb{C}$, $\mathcal{F}(E) = \ell^2$, $\mathcal{T}_+(E) = A(\mathbb{D})$ and $H^\infty(E) = H^\infty(\mathbb{D})$.

2. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $\mathcal{F}(E) = \ell^2(\mathbb{F}_d^+)$, $\mathcal{T}_+(E)$ is Popescu's $\mathcal{A}_d$ and $H^\infty(E)$ is F_d^∞ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts). These algebras are generated by d shifts.

3. M general, $E =_\alpha M$ for an automorphism α. $\mathcal{T}_+(E) = $ the analytic crossed product.

Representations

Theorem

Every completely contractive representation of $\mathcal{T}_+(E)$ on H whose restriction to M is normal, is given by a pair $(\sigma, \mathfrak{z})$ where

1. σ is a normal representation of M on $H = H_\sigma$.
   ($\sigma \in NRep(M)$)

2. $\mathfrak{z} : E \otimes_\sigma H \to H$ is a contraction that satisfies

$$\mathfrak{z}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\mathfrak{z}.$$

We write $\sigma \times \mathfrak{z}$ for the representation and we have $(\sigma \times \mathfrak{z})(\varphi_\infty(a)) = \sigma(a)$ and $(\sigma \times \mathfrak{z})(T_\xi)h = \mathfrak{z}(\xi \otimes h)$ for $a \in M$, $\xi \in E$ and $h \in H$.

Write $\mathcal{I}(\varphi \otimes I, \sigma)$ for the intertwining space and $\mathbb{D}(0, 1, \sigma)$ for the open unit ball there. Thus the c.c. representations of the tensor algebra are parametrized by the family $\{\overline{\mathbb{D}(0, 1, \sigma)}\}_{\sigma \in NRep(M)}$.

Examples

(1) $M = E = \mathbb{C}$. So $\mathcal{T}_+(E) = A(\mathbb{D})$, σ is the trivial representation on H, $E \otimes H = H$ and $\mathbb{D}(0, 1, \sigma)$ is the (open) unit ball in $B(H_\sigma)$.

(2) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathcal{T}_+(E) = \mathcal{A}_d$ (Popescu's algebra) and $\mathbb{D}(0, 1, \sigma)$ is the (open) unit ball in $B(\mathbb{C}^d \otimes H, H)$. Thus the c.c. representations are parameterized by row contractions $(T_1, \ldots, T_d)$.

(3) M general, $E =_\alpha M$ for an automorphism α.
$\mathcal{T}_+(E) =$ the analytic crossed product.
The intertwining space can be identified with
$\{X \in B(H) : \sigma(\alpha(T))X = X\sigma(T), T \in B(H)\}$ and the c.c. representations are $\sigma \times \mathfrak{z}$ where $\mathfrak{z}$ is a contraction there.

Write $\mathcal{I}(\varphi \otimes I, \sigma)$ for the intertwining space

$$\{\mathfrak{z} : E \otimes_\sigma H \to H : \ \mathfrak{z}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\mathfrak{z}.$$

It is a (left) correspondence and its adjoint is written E^σ and is a W^*-correspondence over $\sigma(M)'$. Thus, the representations can be parameterized by

$$\coprod_\sigma (E^\sigma)_1 \subseteq \coprod_\sigma (E^\sigma).$$

When $M = \mathbb{C}$, $E = \mathbb{C}^d$ and σ is the (trivial) representation on an n-dimensional space, $E^\sigma = M_n(\mathbb{C})^d$ and the representations can be parameterized by

$$\coprod_n (M_n(\mathbb{C})^d)_1 \subseteq \coprod_n (M_n(\mathbb{C})^d) = \mathbb{M}^d$$

(an nc set).

Representations of $H^\infty(E)$

The representations of $H^\infty(E)$ are given by the representations of $\mathcal{T}_+(E)$ that extend to an ultraweakly continuous representations of $H^\infty(E)$.

For a given σ, we write $\mathcal{AC}(\sigma)$ for the set of all $\mathfrak{z} \in \overline{\mathbb{D}(0,1,\sigma)}$ such that $\sigma \times \mathfrak{z}$ is a representation of $H^\infty(E)$.

We have

Theorem

$$\mathbb{D}(0,1,\sigma) \subseteq \mathcal{AC}(\sigma) \subseteq \overline{\mathbb{D}(0,1,\sigma)}.$$

Example

When $M = E = \mathbb{C}$, $H^\infty(E) = H^\infty(\mathbb{D})$ and $\mathcal{AC}(\sigma)$ is the set of all contractions in $B(H_\sigma)$ that have an H^∞-functional calculus.

Example

Induced representations: Fix a normal representation π of M on K, let $H = \mathcal{F}(E) \otimes_\pi K$ and define the representation of $H^\infty(E)$ on H by $X \mapsto X \otimes I_K$.

It is $\sigma \times \mathfrak{z}$ for $\sigma(a) = \varphi_\infty(a) \otimes I_K$ and $\mathfrak{z}(\xi \otimes h) = (T_\xi \otimes I_K)h$.

Note that $||\mathfrak{z}|| = 1$ and $\mathfrak{z} \in \mathcal{AC}(\sigma)$.

When π is faithful of infinite multiplicity we write $\sigma_0 \times \mathfrak{s}_0$ for the induced representation. It is essentially independent of π and is a universal generator in the following sense.

Universal induced representation

Theorem

Let $\sigma \times \mathfrak{z}$ be a c.c. representation of $\mathcal{T}_+(E)$ on H. Then the following are equivalent.

(1) *The representation $\sigma \times \mathfrak{z}$ extends to a c.c. ultra weakly continuous representation of $H^\infty(E)$ (that is, $\mathfrak{z} \in \mathcal{AC}(\sigma)$).*

(2) $H = \bigvee\{Ran(C) : C \in \mathcal{I}(\sigma_0 \times \mathfrak{s}_0, \sigma \times \mathfrak{z})\}.$

Here $\mathcal{I}(\sigma_0 \times \mathfrak{s}_0, \sigma \times \mathfrak{z})\}$ is the space of all maps from H_{σ_0} to H_σ that intertwine the representations $\sigma_0 \times \mathfrak{s}_0$ and $\sigma \times \mathfrak{z}$.

Partial results: Douglas (69), Davidson-Li-Pitts (05).

The commutant of an induced representation

Recall that, acting on $H^2(\mathbb{D})$, $H^\infty(\mathbb{D})' = H^\infty(\mathbb{D})$. Also, $H^\infty(\mathbb{D})$ is reflexive.

Similarly, we can view $H^\infty(E)$ as acting on $\mathcal{F}(E) \otimes_\sigma H$ (for a faithful representation σ of M) and write $H^\infty(E^\sigma) \otimes I_H$ for the algebra $H^\infty(E^\sigma)$ represented on $\mathcal{F}(E^\sigma) \otimes_\iota H$ (where ι is the identity representation of $\sigma(M)'$ on H). Then

Theorem

The commutant of $H^\infty(E) \otimes I_H$ is unitarily isomorphic to $H^\infty(E^\sigma) \otimes I_H$.
Consequently (by duality), $(H^\infty(E) \otimes I_H)'' = H^\infty(E) \otimes I_H$.

• In most cases, $H^\infty(E)$ is reflexive (L. Helmer, E. Kakariadis and Bickerton-Kakariadis).

The functions

Conclusion: We now view the elements of $H^\infty(E)$ as functions ($B(H)$-valued) on $\mathcal{AC}(\sigma)$ or on $\mathbb{D}(0,1,\sigma)$. For $F \in H^\infty(E)$, we write $\widehat{F}_\sigma$ for the resulting function. Thus

$$\widehat{F}_\sigma(\mathfrak{z}) = (\sigma \times \mathfrak{z})(F).$$

Note: In fact, for every $F \in H^\infty(E)$, we get a family of functions $\{\widehat{F}_\sigma\}$. The relation between the functions (defined by the same F) for two different σ's) will be discussed later. Now we deal with a fixed σ.

What is the image of this transform? What functions do we get?

Schur class operator functions

Each of the functions $\widehat{F}_\sigma$ ($F \in H^\infty(E)$) is Frechet-differentiable and can be thought of as a (multiple of) generalized Schur class functions.

Recall : The classical Schur class $\mathcal{S}$ consists of the functions f in $H^\infty(\mathbb{D})$ with $\|f\| \leq 1$. The operator valued Schur class $\mathcal{S}(H)$ consists of analytic functions S on $\mathbb{D}$ with $\|S(z)\| \leq 1$ for all $z \in \mathbb{D}$. They have several characterizations. The following is well known.

Theorem

For an $B(H)$-valued function S on $\mathbb{D}$ TFAE:

(1) $S \in \mathcal{S}(H)$.

(2) There is a Hilbert space $\mathcal{E}$ and a coisometric operator

$$U = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) : \left(\begin{array}{c} \mathcal{E} \\ H \end{array} \right) \rightarrow \left(\begin{array}{c} \mathcal{E} \\ H \end{array} \right)$$

so that S can be realized

$$S(z) = D + C(I_{\mathcal{E}} - zA)^{-1}zB.$$

(3)

$$K_S(z, w) = \frac{I - S(z)S(w)^*}{1 - z\overline{w}}$$

is a positive kernel on $\mathbb{D} \times \mathbb{D}$ (into $B(H)$).

• An analogous result holds when one replaces $\mathcal{S}(H)$ by $\{\widehat{F}_\sigma : F \in H^\infty(E), \|F\| < 1\}$.

Theorem

Let E be a W^*-correspondence over M, σ a faithful normal representation of M on H and $Z : \mathbb{D}(0,1,\sigma) \to B(H)$. Then $Z = \hat{F}$ for some $F \in H^\infty(E)$ with $\|F\| \leq 1$ if and only if there is a Hilbert space $\mathcal{E}$, a normal representation τ of $\sigma(M)'$ on $\mathcal{E}$ and a coisometric operator matrix

$$U = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) : \left(\begin{array}{c} \mathcal{E} \\ H \end{array} \right) \to \left(\begin{array}{c} E^\sigma \otimes_\tau \mathcal{E} \\ H \end{array} \right)$$

(with A, B, C, D module maps) so that Z can be realized

$$Z(\mathfrak{z}) = D + C(I_\mathcal{E} - L_\mathfrak{z} A)^{-1} L_\mathfrak{z} B.$$

Here $L_\mathfrak{z} : E^\sigma \otimes_\tau \mathcal{E} \to \mathcal{E}$ is defined by $L_\mathfrak{z}(\eta \otimes h) = \mathfrak{z}\eta h$.

Theorem

Let E be a W^*-correspondence over M, σ a faithful normal representation of M on H and $Z : \mathbb{D}(0, 1, \sigma) \to B(H)$. Then $Z = \hat{F}$ for some $F \in H^\infty(E)$ with $\|X\| \leq 1$ if and only if the kernel $K_Z : \mathbb{D}(0, 1, \sigma) \times \mathbb{D}(0, 1, \sigma) \to B(\sigma(M)', B(H))$ is completely positive definite (BBLS04) where

$$K_Z(\mathfrak{z}, \mathfrak{v}) = (id - Ad(Z(\mathfrak{z}), Z(\mathfrak{v})) \circ (id - \theta_{\mathfrak{z}, \mathfrak{v}})^{-1}.$$

Here $Ad(Z(\mathfrak{z}), Z(\mathfrak{v}))(a) = Z(\mathfrak{z})aZ(\mathfrak{v})^*$ and $\theta_{\mathfrak{z}, \mathfrak{v}}(a) = \mathfrak{z}(I_E \otimes a)\mathfrak{v}^*$ for $a \in \sigma(M)'$. The complete positivity of K_Z means that, for every $\mathfrak{z}_1, \ldots, \mathfrak{z}_m \in \mathbb{D}(0, 1, \sigma)$, the matrix of maps $(K_Z(\mathfrak{z}_i, \mathfrak{z}_j))$ defines a completely positive map from $M_m(\sigma(M)')$ into $M_m(B(H))$.

Automorphisms

Viewing $H^\infty(E)$ as an algebra of functions on $\mathbb{D}(0,1,\sigma)$, one can expect to have a relationship between automorphisms of this algebra and appropriate automorphisms on the domain. Given an (completely isometric, w^*-homeomorphic) automorphism α of $H^\infty(E)$ and a fixed representation σ of M, it is clear how to define the map τ of the domain :

$$\widehat{T_\xi}(\tau(\mathfrak{z})) = \widehat{\alpha(T_\xi)}(\mathfrak{z}), \ \xi \in E.$$

Thus

$$\tau(\mathfrak{z})(\xi \otimes h) = \widehat{\alpha(T_\xi)}(\mathfrak{z})h, \ h \in H.$$

Then τ maps $\mathbb{D}(0,1,\sigma)$ into $\mathcal{AC}(\sigma)$ and, under certain assumptions, into itself.

Sample results :

Theorem

If $Z(M)$, the center of M, is atomic and the left action of M on E is faithful, and α is an automorphism that leaves $\varphi_\infty(M)$ elementwise fixed, then τ is a biholomorphic map from $\mathbb{D}(0, 1, \sigma)$ onto itself and, for every $X \in H^\infty(E)$ and $\mathfrak{z} \in \mathbb{D}(0, 1, \sigma)$,

$$\widehat{\alpha(X)}(\mathfrak{z}) = \widehat{X}(\tau(\mathfrak{z})).$$

Definition

The centre of $\mathbb{D}(0, 1, \sigma)$ is

$$Z\mathbb{D}(0, 1, \sigma) := \{\mathfrak{z} \in \mathbb{D}(0, 1, \sigma) : c\mathfrak{z} = \mathfrak{z}(I_E \otimes c), \ \ c \in \sigma(M)'\}.$$

Fact : A map τ as above would map $Z\mathbb{D}(0, 1, \sigma)$ into itself.

Lemma

Given $\gamma \in Z\mathbb{D}(0, 1, \sigma)$, one can define a biholomorphic map g_γ of $\mathbb{D}(0, 1, \sigma)$ that maps 0 to γ and γ to 0 (a "Moebius transformation") and an automorphism α_γ of $H^\infty(E)$ such that

$$\widehat{\alpha_\gamma(F)}(\mathfrak{z}) = \widehat{F}(g_\gamma(\mathfrak{z})).$$

Note: $Z\mathbb{D}(0, 1, \sigma)$ is the orbit of 0 under the maps τ associated with automorphisms of $H^\infty(E)$ as above.

Theorem

Let E be a faithful W^*-correspondence over M and σ be a faithful representation of M. Let α be an automorphism of $H^\infty(E)$ that leaves $\varphi_\infty(M)$ elementwise fixed and for which a map τ, as above, exists (in particular, if $Z(M)$ is atomic). Then there is some $\gamma \in Z\mathbb{D}(0, 1, \sigma)$ and a unitary operator u in $\mathcal{L}(E)$, satisfying $u(Z(E)) = Z(E)$, such that

$$\alpha = \alpha_\gamma \circ \alpha_u,$$

where $\alpha_u(T_\xi) = T_{u\xi}$ for every $\xi \in E$ and α_γ is as in the lemma. In particular, if $Z(E) = \{0\}$, every such automorphism is α_u for some unitary operator $u \in \mathcal{L}(E)$.

The families of functions

Recall that, given $F \in H^\infty(E)$, we define a family $\{\widehat{F}_\sigma\}_{\sigma \in NRep(M)}$ of (operator valued) functions by

$$\widehat{F}_\sigma(\mathfrak{z}) = (\sigma \times \mathfrak{z})(F)$$

(defined on $\mathcal{AC}(\sigma)$ or on $\mathbb{D}(0, 1, \sigma)$ and takes values in $B(H_\sigma)$).
Here $NRep(M)$ is the set of all normal representations of M.
Note that the family of domains (either $\{\mathcal{AC}(\sigma)\}$ or $\{\mathbb{D}(0, 1, \sigma)\}$) is a matricial family in the following sense.

Definition

A family of sets $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$, with $\mathcal{U}(\sigma) \subseteq \mathcal{I}(\varphi \otimes I, \sigma)$, satisfying $\mathcal{U}(\sigma) \oplus \mathcal{U}(\tau) \subseteq \mathcal{U}(\sigma \oplus \tau)$ is called a *matricial family* of sets.

Definition

Suppose $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$ is a matricial family of sets and suppose that for each $\sigma \in NRep(M)$, $f_\sigma : \mathcal{U}(\sigma) \to B(H_\sigma)$ is a function. We say that $f := \{f_\sigma\}_{\sigma \in NRep(M)}$ is a *matricial family of functions* in case

$$Cf_\sigma(\mathfrak{z}) = f_\tau(\mathfrak{w})C \tag{1}$$

for every $\mathfrak{z} \in \mathcal{U}(\sigma)$, every $\mathfrak{w} \in \mathcal{U}(\tau)$ and every $C \in \mathcal{I}(\sigma \times \mathfrak{z}, \tau \times \mathfrak{w})$ (equivalently, $C \in \mathcal{I}(\sigma, \tau)$ and $C\mathfrak{z} = \mathfrak{w}(I_E \otimes C)$).

Theorem

For every $F \in H^\infty(E)$, the family $\{\widehat{F}_\sigma\}$ is is a matricial family (on $\{\mathcal{AC}(\sigma)\}$).

Conversely, if $f = \{f_\sigma\}_{\sigma \in NRep(M)}$ is a matricial family of functions, with f_σ defined on $\mathcal{AC}(\sigma)$ and mapping to $B(H_\sigma)$, then there is an $F \in H^\infty(E)$ such that f is the **Berezin transform** of F, i.e., $f_\sigma = \widehat{F}_\sigma$ for every σ.

Notation: For $\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma)$ and $k \geq 1$,
$\mathfrak{z}^{(k)} = \mathfrak{z}(I_E \otimes \mathfrak{z}) \cdots (I_{E^{\otimes k-1}} \otimes \mathfrak{z}) \in \mathcal{I}(\varphi_{E^{\otimes k}} \otimes I, \sigma)$.
For a sequence $\theta = \{\theta_k\}$, with $\theta_k \in E^{\otimes k}$,

$$L_{\theta_k} : H \to E^{\otimes k} \otimes H, \quad L_{\theta_k} h = \theta_k \otimes h$$

and $R(\theta) = (\limsup_{k \to \infty} \|\theta_k\|^{\frac{1}{k}})^{-1}$. (Popescu)

Theorem

*If $f = \{f_\sigma\}_{\sigma \in MRep(M)}$ is a family of functions, with f_σ mapping $\mathbb{D}(0, 1, \sigma)$ to $B(H_\sigma)$, then f is a matricial family of functions if and only if there is a formal tensor series θ with $R(\theta) \geq 1$ such that f is the family of **tensorial power series** determined by θ; that is,*

$$f_\sigma(\mathfrak{z}) = \sum_{k \geq 0} \mathfrak{z}^{(k)} L_{\theta_k}.$$

Moreover, $f = \widehat{F}$ for some $F \in H^\infty(E)$ if and only if

$$\sup\{\|f_\sigma(\mathfrak{z})\| \mid \sigma \in NRep(M), \ \mathfrak{z} \in \mathbb{D}(0, 1, \sigma)\} < \infty. \quad (2)$$

Function theory without a generator

Now we fix an additive subcategory Σ of $NRep(M)$ that do not necessarily contain a special generator. Then

Theorem

Suppose that $f = \{f_\sigma\}_{\sigma \in \Sigma}$ is a matricial family of functions defined on $\{\mathbb{D}(0, 1, \sigma)\}$ that is locally uniformly bounded in the sense that for each $r < 1$, $\sup_{\sigma \in \Sigma} \sup_{\mathfrak{z} \in \mathbb{D}(0,r,\sigma)} \|f_\sigma(\mathfrak{z})\| < \infty$. Then:

1. *Each f_σ is Frechet analytic on $\mathbb{D}(0, 1, \sigma)$ and*

$$f_\sigma(\mathfrak{z}) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f_\sigma(0)(\mathfrak{z}).$$

2. *If the subcategory is full and if each $\sigma \in \Sigma$ is faithful, then there is $\theta = \{\theta_k\}$ with $R(\theta) \geq 1$ and*

$$f_\sigma(\mathfrak{z}) = \sum_{k \geq 0} \mathcal{Z}_k(\mathfrak{z}) L_{\theta_k}$$

Now, inspired by V. Vinnikov and D.Kalyuznyi-Verbovetsky, we discuss another expansion: the Taylor-Taylor series. Generalizing their analysis, one can define $\Delta^n f_\sigma$ - the nth- order Taylor derivative of f_σ. We get the following.

Theorem (T-T Series)

Let $f = \{f_\sigma\}_{\sigma \in \Sigma}$ be a matricial family of functions defined on a matricial disc $\mathbb{D}(0, r)$ $(= \{\mathbb{D}(0, r, \sigma)\}_\sigma)$ and suppose that f is locally uniformly bounded. Then:

1. *Each f_σ is Frechet differentiable in $\mathfrak{z}$, $\mathfrak{z} \in \mathbb{D}(0, r, \sigma)$, and*

$$f'_\sigma(\mathfrak{z})(\mathfrak{w}) = \Delta f(\mathfrak{z})(\mathfrak{w}).$$

2.

$$D^k f_\sigma(0)(\mathfrak{w}) = k! \Delta^k f_\sigma(0)(\mathfrak{w}).$$

3. *Each f_σ may be expanded on $\mathbb{D}(0, r, \sigma)$ as*

$$f_\sigma(\mathfrak{z}) = \sum_{k=0}^{\infty} \Delta^k f_\sigma(0)(\mathfrak{z}, \dots, \mathfrak{z}), \tag{3}$$

where the series converges absolutely and uniformly on every disc $\mathbb{D}(0, r_0, \sigma)$ with $r_0 < r$.

Other domains

Q: Can we construct other operator algebras (associated with E) whose representations will be parameterized by other matricial families of subsets of $\coprod_\sigma \mathcal{I}(\varphi \otimes I, \sigma)$?

Inspired by the works of V. Muller and G. Popescu, we studied algebras of weighted shifts.

To define the general situation, let $Z = \{Z_k\}$ such that

- $Z_k \in \mathcal{L}(E^{\otimes k}) \cap \varphi_k(M)'$.
- $Z_k \geq 0$ and invertible for all $k \geq 1$.
- $\sup_k \|Z_k\| < \infty$

and define, for $\xi \in E$, the Z-weighted shift $W_\xi \in \mathcal{L}(\mathcal{F}(E))$ by

$$W_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = Z_{n+1}(\xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n).$$

and $W_\xi b = Z_1(\xi b)$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{W_\xi : \xi \in E\}$ will be called the **Z-tensor algebra** of E and denoted $\mathcal{T}_+(E, Z)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E, Z)$ will be called the **Z-Hardy algebra** of E and denoted $H^\infty(E, Z)$.

Q: What are the representations of these algebras? In general: Unknown.

Theorem

Under certain conditions the representations of the Z-tensor algebra are parameterized by $\coprod_\sigma \overline{D}_{X,\sigma}$ where

$$\overline{D}_{X,\sigma} := \{\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma) : \quad ||\sum_{k=1}^\infty \mathfrak{z}^{(k)}(X_k \otimes I_{H_\sigma})\mathfrak{z}^{(k)*}|| \leq 1\}$$

and $X = \{X_k\}$ is a sequence defined by Z.

Examples

- If $E = M = \mathbb{C}$, σ is on H and $X_k = x_k \in \mathbb{C}$,
  $\overline{D}_{X,\sigma} = \{T \in B(H) : \sum_k x_k T^k T^{*k} \leq I\}$.

- If $M = \mathbb{C}$, $E = \mathbb{C}^d$, σ is on H and X_k is the $d^k \times d^k$ matrix $(x_{\alpha,\beta})$ (where α, β are words of length k in $\{1, \ldots, d\}$),

$$\overline{D}_{X,\sigma} = \{T = (T_1, \ldots, T_d) : \sum_{|\alpha|=|\beta|} x_{\alpha,\beta} T_\alpha T_\beta^* \leq I\}$$

  where $T_\alpha = T_{\alpha_1} \cdots T_{\alpha_k}$.

- If $E =_\alpha M$, $x_k \in Z(M)$ and

$$\overline{D}_{x,\sigma} = \{T \in B(H_\sigma) : T\sigma(\alpha(\cdot)) = \sigma(\cdot)T, \sum_k T^k \sigma(x_k) T^{k*} \leq I\}.$$

Theorem

For every $F \in H^\infty(E, Z)$, the family $\{\widehat{F}_\sigma\}$ is a matricial family of functions on $\{D_{X,\sigma}\}_\sigma$ and on $\{\mathcal{AC}(\sigma)\}$.

Does the converse hold?

Theorem

If $f = \{f_\sigma\}_{\sigma \in NRep(M)}$ is a matricial family of functions, with f_σ defined on $\mathcal{AC}(\sigma)$ and mapping to $B(H_\sigma)$, then there is an $F \in H^\infty(E, Z)$ such that f and the Berezin transform of F, $\widehat{F}$, agree on $D_{X,\sigma}$, i.e.,

$$f_\sigma(\mathfrak{z}) = \widehat{F}_\sigma(\mathfrak{z})$$

for every σ and every $\mathfrak{z} \in D_{X,\sigma}$.

Another thought:

Together with O. Shalit, we studied Hardy (and tensor) algebras associated with subproduct systems. Some of the results (such as dilation results) there (obtained by us and by A. Viselter) bear strong similarity to results on the Hardy algebras of weighted shifts. As an example, recall that, the representations of $H^\infty(E, Z)$ are given by

$$\overline{D}_{X,\sigma} := \{\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma) : \quad ||\sum_{k=1}^\infty \mathfrak{z}^{(k)}(X_k \otimes I_{H_\sigma})\mathfrak{z}^{(k)*}|| \le 1\}.$$

For a subproduct tensor algebra (associated with the subproduct $\{Y(n)\}$ where $Y(n) = p_n E^{\otimes n}$ the representations are given by

$$\{\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma) : \quad ||\sum_{k=1}^\infty \mathfrak{z}^{(k)}(X_k \otimes I_{H_\sigma})\mathfrak{z}^{(k)*}|| \le 1\}$$

(where $X_1 = I$, $X_k = \infty p_k^\perp$ $k > 1$: the sequence one gets from $Z_k = p_k$) which is a (strange) way of saying: $\mathfrak{z}^{(k)}|p_n^\perp = 0$ for all k and $||\mathfrak{z}|| \le 1$.

There should be a way to formalize this similarity.

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Thank You !