Cuntz-Pimsner algebras from group representations

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Multivariable Operator Theory

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Outline

- We discuss various constructions of C^*-correspondences arising from group representations and from group actions on graphs.
- The class of the associated Cuntz-Pimsner algebras is very large and in some cases we can identify them as graph algebras or crossed products and determine their K-theory.
- There are connections with Doplicher-Roberts algebras, Nekrashevych algebras of self-similar group actions and with Cuntz-Pimsner algebras considered by Kumjian.
- We illustrate with several examples.

C^*-correspondences

- Let A, B be C^*-algebras. A right Hilbert B-module $\mathcal{X}$ is a Banach space with a right action of B and an B-valued inner product satisfying

$$\langle \xi, \eta b \rangle = \langle \xi, \eta \rangle b, \;\; \langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*,$$

$$\langle \xi, \xi \rangle \geq 0, \text{ and } ||\xi|| = ||\langle \xi, \xi \rangle||^{1/2}.$$

- We say that $\mathcal{X}$ is a A–B C^*-correspondence if moreover there is a $*$-homomorphism $\phi : A \to \mathcal{L}_B(\mathcal{X})$ which gives the left action, where $\mathcal{L}_B(\mathcal{X})$ denotes the C*-algebra of all adjointable operators on $\mathcal{X}$.
- Suppose $A = B$. A *Toeplitz representation* of the C^*-correspondence $\mathcal{X}$ over A in a C^*-algebra D is a pair (τ, π) with $\tau : \mathcal{X} \to D$ a linear map and $\pi : A \to D$ a $*$-homomorphism, such that

$$\tau(\xi \cdot a) = \tau(\xi)\pi(a), \; \tau(\xi)^*\tau(\eta) = \pi(\langle \xi, \eta \rangle)$$

$$\tau(a \cdot \xi) = \pi(a)\tau(\xi).$$

Cuntz-Pimsner algebras

- The universal C^*-algebra for such representations is called the Toeplitz algebra of $\mathcal{X}$, denoted by $\mathcal{T}_{\mathcal{X}}$.
- There is a $*$-homomorphism $\psi : \mathcal{K}_A(\mathcal{X}) \to D$ such that

$$\psi(\theta_{\xi,\eta}) = \tau(\xi)\tau(\eta)^*,$$

  where $\mathcal{K}_A(\mathcal{X})$ is the closed linear span of the operators $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle$.
- The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ is universal for Toeplitz representations which are Cuntz-Pimsner covariant:

$$\psi(\phi(a)) = \pi(a) \text{ for all } a \in J_{\mathcal{X}} := \phi^{-1}(\mathcal{K}_A(\mathcal{X})) \cap (\ker\phi)^{\perp}.$$

C^*-algebras of graphs

- Let $E = (E^0, E^1, r, s)$ be a directed graph with E^0 and E^1 at most countable.
- If $r^{-1}(v)$ is finite for all v, the algebra $C^*(E)$ is defined using projections p_v for $v \in E^0$ and partial isometries u_e for $e \in E^1$ with

$$u_e^* u_e = p_{s(e)} \text{ for } \ e \in E^1, \ \ p_v = \sum_{r(e)=v} u_e u_e^* \ \text{ for } \ v \in r(E^1).$$

- We view $C^*(E)$ as the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ of the C^*-correspondence $\mathcal{X} = \mathcal{X}_E$ over $A = C_0(E^0)$, obtained as a completion of $C_c(E^1)$ with the inner product

$$\langle \xi, \eta \rangle (v) = \sum_{s(e)=v} \overline{\xi(e)} \eta(e), \ \xi, \eta \in C_c(E^1)$$

and multiplications

$$(\xi \cdot f)(e) = \xi(e) f(s(e)), \ (f \cdot \xi)(e) = f(r(e)) \xi(e).$$

C^*-correspondences from representations

- Let G be a locally compact group and let $\rho : G \to U(\mathcal{H})$ be a unitary representation.
- Let $\pi = \pi_\rho : C^*(G) \to \mathcal{L}(\mathcal{H})$ be the extension of ρ to the group C^*-algebra,

$$\pi(f)\xi = \int_G f(t)\rho(t)\xi dt \ \text{ for } f \in L^1(G), \xi \in \mathcal{H}.$$

- Then $\mathcal{E} = \mathcal{E}(\rho) = \mathcal{H} \otimes_{\mathbb{C}} C^*(G)$ becomes a C^*-correspondence over $C^*(G)$ with inner product

$$\langle \xi \otimes a, \eta \otimes b \rangle = \langle \xi, \eta \rangle a^* b$$

and operations

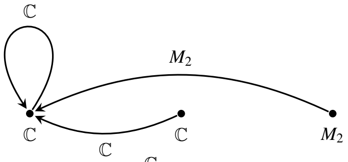
$$(\xi \otimes a) \cdot b = \xi \otimes ab, \ \ a \cdot (\xi \otimes b) = \pi(a)\xi \otimes b.$$

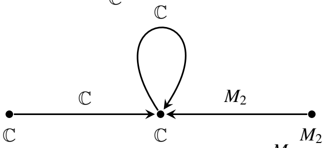
A first result

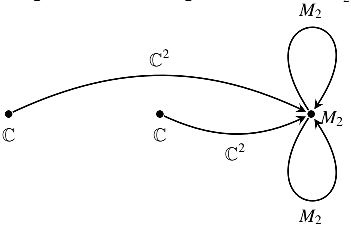
- **Theorem** (D). If G is a compact group and $\rho : G \to U(\mathcal{H})$ is any representation with $\mathcal{H}$ separable, then $\mathcal{O}_{\mathcal{E}(\rho)}$ is SME to a graph C^*-algebra.
  If $\pi_\rho : C^*(G) \to \mathcal{L}(\mathcal{H})$ is injective, then the graph has no sources.
  If $\rho \cong \rho_1 \oplus \rho_2$, then the incidence matrix for the graph of ρ is the sum of incidence matrices for ρ_1 and ρ_2.

- **Proof** (sketch). The group C^*-algebra $C^*(G)$ decomposes as a direct sum of matrix algebras A_i with units p_i, indexed by the discrete set $\hat{G}$.
- Let E be the graph with vertex space $E^0 = \hat{G}$ and with edges determined by the A_j-A_i C^*-correspondences $p_j\mathcal{E}(\rho)p_i$.
- $\mathcal{O}_{\mathcal{E}(\rho)}$ is isomorphic to the C^*-algebra of a graph of C^*-correspondences in which we assign the algebra A_i at the vertex v_i and the minimal components of $p_j\mathcal{E}(\rho)p_i$ for each edge from v_i to v_j.
- By construction, this C^*-algebra is SME to $C^*(E)$.

Example: S_3

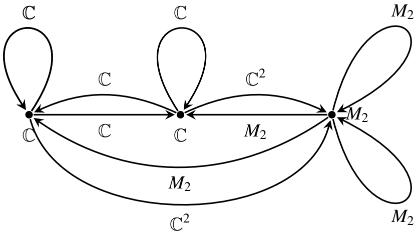
- Donote by S_3 the symmetric group. Then $\hat{S_3} = \{\iota, \varepsilon, \sigma\}$ and the graphs associated with the representations ι, ε and σ are
- 

- 

- 

Example: S_3

- For $\rho = \sigma \otimes \sigma$ we get the following graph of C^*-correspondences



with incidence matrix

$$B_{\sigma\otimes\sigma} = B_\iota + B_\varepsilon + B_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Example: cyclic groups

- Any representation ρ of a cyclic group G is determined by a unitary $\rho(1) \in U(\mathcal{H})$ that decomposes into characters.
- If $G = \mathbb{Z}/n\mathbb{Z}$, then $\hat{G} = \{\chi_1, ..., \chi_n\}$ and $\mathcal{E}(\rho)$ determines a graph with n vertices and incidence matrix $[a_{ji}]$, where $a_{ji} = \dim \chi_j \mathcal{H} \chi_i$.
- For $G = \mathbb{Z}$, assume that $\mathcal{H} = L^2(X, \mu)$ for a measure space (X, μ) and that $\rho(1) = M_\varphi$, the multiplication operator with a function $\varphi : X \to \mathbb{T}$.
- Then $\mathcal{E}(\rho) = L^2(X, \mu) \otimes C^*(\mathbb{Z}) \cong C(\mathbb{T}, L^2(X, \mu))$ becomes a C^*-correspondence over $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ with operations

$$\langle \xi, \eta \rangle(k) = \langle \xi(k), \eta(k) \rangle_{L^2(X,\mu)}, \quad \text{for} \;\; \xi, \eta \in C_c(\mathbb{Z}, L^2(X, \mu))$$

and

$$(\xi \cdot f)(k) = \sum_k \xi(k) f(k), \quad (f \cdot \xi)(k) = \sum_k f(k) \varphi^k \xi(k),$$

$$\text{for} \;\; f \in C_c(\mathbb{Z}), \xi \in C_c(\mathbb{Z}, L^2(X, \mu)).$$

Example: cyclic groups

- If $\dim L^2(X,\mu) = n$ is finite, then the function φ is given by $(w_1, ..., w_n) \in \mathbb{T}^n$ and $\mathcal{E}(\rho)$ is isomorphic to the C^*-correspondence of the following topological graph.
- The vertex space is $E^0 = \mathbb{T}$, the edge space is $E^1 = \mathbb{T} \times \{1, 2, ..., n\}$, and the source and range maps are

$$s : E^1 \to E^0, s(z,k) = z, \quad r : E^1 \to E^0, r(z,k) = w_k z.$$

- If λ is the left regular representation of $G = \mathbb{Z}$ on $\ell^2(\mathbb{Z})$, we will see that $\mathcal{O}_{\mathcal{E}(\lambda)}$ is simple and purely infinite with the same K-theory as $C(\mathbb{T})$.

Kumjian's construction

- **Theorem** (Kumjian). Let A be a separable unital C^*-algebra and let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$.
- Then $\mathcal{E} = \mathcal{H} \otimes_{\mathbb{C}} A$ is a C^*-correspondence such that $\mathcal{O}_{\mathcal{E}} \cong \mathcal{T}_{\mathcal{E}}$ is simple, purely infinite and KK-equivalent to A.
- **Corollary**. Let G be an infinite, discrete and amenable group, and let $\lambda : G \to U(\ell^2(G))$ be the left regular representation. Then $\mathcal{O}_{\mathcal{E}}$ is simple and purely infinite, KK-equivalent to $C^*(G)$.
- **Proof**. Since G is amenable, the representation $\pi_\lambda : C^*(G) \to \mathcal{L}(\ell^2(G))$ induced by λ is faithful. Since G is infinite,

$$\pi_\lambda(C^*(G)) \cap \mathcal{K}(\ell^2(G)) = \{0\}.$$

Example: $\mathbb{R}$

- Let $G = \mathbb{R}$ and let μ be the Lebesgue measure on $\mathbb{R}$.
- Consider the representation

$$\rho : \mathbb{R} \to U(L^2(\mathbb{R},\mu)), (\rho(t)\xi)(s) = e^{its}\xi(s)$$

which extends to the Fourier transform π_ρ on $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$, where

$$(\pi_\rho(f)\xi)(s) = \int_{\mathbb{R}} f(t)e^{its}\xi(s)d\mu(t), \quad \text{for } f \in L^1(\mathbb{R},\mu), \xi \in L^2(\mathbb{R},\mu).$$

- It is known that ρ is equivalent to the right regular representation of $\mathbb{R}$ and that ρ is a direct integral of characters χ_t, where $\chi_t(s) = e^{its}$.
- Then $\mathcal{E}(\rho) = L^2(\mathbb{R},\mu) \otimes C^*(\mathbb{R}) \cong C_0(\mathbb{R}, L^2(\mathbb{R},\mu))$ becomes a C^*-correspondence over $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ such that the left multiplication is injective and $\pi_\rho(C_0(\mathbb{R})) \cap \mathcal{K}(L^2(\mathbb{R},\mu)) = \{0\}$.
- It follows that $\mathcal{O}_{\mathcal{E}(\rho)}$ has the K-theory of $C_0(\mathbb{R})$, but since $C_0(\mathbb{R})$ is not unital, we cannot apply Kumjian's Theorem to conclude that this algebra is simple or purely infinite.

The crossed product C^*-correspondence

- Let G be locally compact and let $\rho : G \to U(\mathcal{H})$ be a representation with $\dim \mathcal{H} = n \in \{1, 2, 3, ...\} \cup \{\infty\}$.
- Then $\mathcal{D}(\rho) = \mathcal{H} \otimes_{\mathbb{C}} C^*(G)$ with the same inner product and right multiplication as $\mathcal{E}(\rho)$ becomes a C^*-correspondence using the left multiplication
$$(h \cdot \xi)(t) = \int_G h(s)\rho(s)\xi(s^{-1}t)ds$$
for $\xi \in C_c(G, \mathcal{H})$ and $h \in C_c(G)$.
- This left multiplication is always injective and $\mathcal{D}(\rho)$ is nondegenerate.
- **Theorem**. The representation ρ determines a quasi-free action of G on the Cuntz algebra $\mathcal{O}_n$ such that $\mathcal{O}_{\mathcal{D}(\rho)} \cong \mathcal{O}_n \rtimes_\rho G$.

The abelian case

- If G is compact and abelian, then ρ decomposes into characters and determines a cocycle $c : E_n^1 \to \hat{G}$, where E_n is the graph with one vertex and n edges.
- $\mathcal{O}_n \rtimes_\rho G$ is isomorphic to $C^*(E_n(c))$, where $E_n(c)$ is the skew product graph $(\hat{G}, \hat{G} \times E_n^1, r, s)$ with

$$r(\chi, e) = \chi c(e), s(\chi, e) = \chi.$$

- If $G = \mathbb{T}$ and $\lambda : \mathbb{T} \to U(L^2(\mathbb{T}))$ is the left regular representation, then $\mathcal{O}_\infty \rtimes_\lambda \mathbb{T}$ is isomorphic to the graph algebra where the vertices are labeled by $\mathbb{Z}$ and the incidence matrix has each entry equal to 1.
- If $G = \mathbb{R}$ and n is finite, Kishimoto and Kumjian showed that $\mathcal{O}_n \rtimes_\rho \mathbb{R}$ is simple and purely infinite if the corresponding characters generate $\mathbb{R}$ as a closed semigroup.

Group actions on C^*-correspondences

- A group action on a C^*-correspondence $\mathcal{X}$ over A is a homomorphism $\rho : G \to \mathcal{L}_{\mathbb{C}}(\mathcal{X})$ with values invertible $\mathbb{C}$-linear operators on $\mathcal{X}$ and an action of G on A by $*$-automorphisms such that

$$\langle \rho(g)\xi, \rho(g)\eta \rangle = g \cdot \langle \xi, \eta \rangle,$$

$$\rho(g)(\xi a) = (\rho(g)\xi)(g \cdot a),\ \rho(g)(a \cdot \xi) = (g \cdot a)(\rho(g)\xi).$$

- A group action on a graph E determines an action on $\mathcal{X}_E$ by

$$(\rho(g)\xi)(e) = \xi(g^{-1} \cdot e) \ \text{ for } \xi \in C_c(E^1),$$

$$(g \cdot a)(v) = a(g^{-1} \cdot v) \ \text{ for } a \in C_0(E^0).$$

- By the universal property, an action of G on $\mathcal{X}_E$ determines an action of G on $C^*(E)$ and an action on the core AF-algebra $C^*(E)^{\mathbb{T}}$.

Crossed products

- Let A be a C^*-algebra and let $\mathcal{X}$ be a C^*-correspondence over A. An action of G on $\mathcal{X}$ determines an action on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$.
- The crossed product $\mathcal{X} \rtimes G = \mathcal{X} \otimes_A (A \rtimes G)$ becomes a C^*-correspondence over $A \rtimes G$ after the completion of $C_c(G, \mathcal{X})$ using the operations

$$\langle \xi, \eta \rangle(t) = \int_G s^{-1} \cdot \langle \xi(s), \eta(st) \rangle ds,$$

$$(\xi \cdot f)(t) = \int_G \xi(s)(s \cdot f(s^{-1}t))ds, \ (f \cdot \xi)(t) = \int_G f(s) \cdot (s \cdot \xi(s^{-1}t))ds,$$

where $\xi \in C_c(G, \mathcal{X})$,$f \in C_c(G, A)$. Recall

- **Theorem** (Hao, Ng). If G is amenable, then

$$\mathcal{O}_{\mathcal{X} \rtimes G} \cong \mathcal{O}_{\mathcal{X}} \rtimes G.$$

Another result

- **Theorem** (D). Given a discrete locally finite graph E and a finite group G acting on E, the crossed product $C^*(E) \rtimes G$ is the C^*-algebra of a graph of minimal C^*-correspondences.
- Hence $C^*(E) \rtimes G$ is SME to a graph C^*-algebra, where the number of vertices is the cardinality of the spectrum of $C_0(E^0) \rtimes G$.
- **Proof** (sketch). Recall that if G acts on a finite or countable set X, then $C_0(X) \rtimes G$ decomposes as a direct sum of crossed products $C(Gx) \rtimes G$ over the orbit space X/G.
- Since the action on each orbit Gx is transitive, this can be identified with G/G_x, where G_x is the stabilizer group and G acts on G/G_x by left multiplication. Moreover,

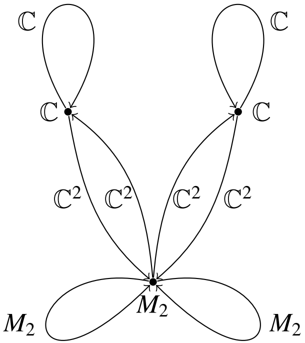
$$C(G/G_x) \rtimes G \cong M_{|Gx|} \otimes C^*(G_x).$$

Sketch of Proof

- We decompose the C^*-correspondence $\mathcal{X}_E \rtimes G$ over the C^*-algebra $C_0(E^0) \rtimes G$.
- Let $C_0(E^0) \rtimes G \cong \bigoplus_{i=1}^{n} A_i$, where $n \in \mathbb{N} \cup \{\infty\}$ and A_i are simple matrix algebras. This decomposition is obtained in two stages, from the orbits in E^0 and from the characters of the stabilizer groups.
- Consider the graph with n vertices and at each vertex v_i we assign the C^*-algebra A_i.
- The edges and the assigned C^*-correspondences are constructed from the orbits in E^1 and multiplicities.
- $C^*(E) \rtimes G$ is isomorphic to the C^*-algebra of this graph of (minimal) C^*-correspondences.
- **Corollary**. We have $C^*(E)^{\mathbb{T}} \rtimes G \cong (C^*(E) \rtimes G)^{\mathbb{T}}$.

Example

- For the permutation action of S_3 on the $\mathcal{O}_3$ graph we get the following graph of C^*-correspondences for $\mathcal{O}_3 \rtimes S_3$:



Exel-Pardo C^*-correspondences

- Let G be a group acting on a graph $E = (E^0, E^1, r, s)$. Recall that G acts on $A = C_0(E^0)$ and on $\mathcal{X} = \mathcal{X}_E$ by

$$\alpha_g(f)(v) = f(g^{-1}v), \;\; f \in C_0(E^0), \;\; \gamma_g(\xi)(e) = \xi(g^{-1}e), \;\; \xi \in C_c(E^1).$$

- A cocycle is a map $\varphi : G \times E^1 \to G$ such that $\varphi(g,e) \cdot s(e) = g \cdot s(e)$ for all $(g,e) \in G \times E^1$. Particular case $\varphi(g,e) = g$.
- Define an action of G on the Hilbert module $\mathcal{X}_E \rtimes_\gamma G$ by

$$(V_g\xi)(e,h) = \xi(g^{-1}e, \varphi(g^{-1},e)h), \;\; \xi \in C_c(E^1 \times G).$$

- Together with the left action of $C_0(E^0)$ on $\mathcal{X}_E \rtimes_\gamma G$ given by

$$(\pi(f)\xi)(e,h) = f(r(e))\xi(e,h),$$

  we get a covariant representation (π, V) of $(C_0(E^0), G, \alpha)$.
- This induces a map $C_0(E^0) \rtimes_\alpha G \to \mathcal{L}(\mathcal{X}_E \rtimes_\gamma G)$, so $\mathcal{X}_E \rtimes_\gamma G$ becomes a C^*-correspondence $\mathcal{X}_E(\varphi)$ over $C_0(E^0) \rtimes_\alpha G$.
- The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}_E(\varphi)}$ is the Exel-Pardo algebra associated to (E, G, φ).

Example: Katsura algebras

- To realize all Kirchberg algebras as topological graph algebras and to lift automorphisms of K-theory groups, Katsura introduced the algebras $\mathcal{O}_{A,B}$ for certain $n \times n$ matrices A, B with integer entries.
- Using generators and relations, Exel and Pardo proved that $\mathcal{O}_{A,B} \cong \mathcal{O}_{\mathcal{X}_E(\varphi)}$, where
- The matrix A is the incidence matrix of a finite graph E and the matrix B defines an action of $\mathbb{Z}$ on E.
- This action fixes the vertices and if the edges from i to j are labeled e_{ijn} for $0 \leq n < A_{ij}$, then $m \cdot e_{ijn} = e_{ijr}$, where $mB_{ij} + n = qA_{ij} + r$ with $0 \leq r < A_{ij}$.
- The cocycle is $\varphi : \mathbb{Z} \times E^1 \to \mathbb{Z}, \;\; \varphi(m, e_{ijn}) = q$.

Self-similar actions

- Let X be a finite alphabet with n letters and let $X^* = \bigcup_{k=0}^{\infty} X^k$ be the set of finite words with $X^0 = \{\emptyset\}$.
- An action of G on X^* is self-similar if for all $g \in G$ and $x \in X$ there exist unique $y \in X$ and $h \in G$ such that

$$g \cdot (xw) = y(h \cdot w)$$

  for all $w \in X^*$.
- A self-similar action comes from an action of G on the graph E_n with one vertex and n loops.
- The cocycle is given by $\varphi(g, e) = h$, where $g \cdot (xe) = y(h \cdot e)$, since X^* is the space of finite paths in E_n.
- Let T_X be the universal cover of E_n, which is a directed tree. We obtain
- **Theorem**. There is an action of G on T_X and on the boundary ∂T_X such that $C^*(T_X) \rtimes G$ is SME with $C_0(\partial T_X) \rtimes G$.

Examples

- **The odometer**. Let $X = \{0, 1\}$ and $G = \mathbb{Z} = \langle a \rangle$ where

$$a \cdot (0w) = 1w, \ \ a \cdot (1w) = 0(a \cdot w), w \in X^*.$$

- It is known that ∂T_X is a Cantor set, the action of $\mathbb{Z}$ on ∂T_X is minimal and $C_0(\partial T_X) \rtimes \mathbb{Z}$ is the Bunce-Deddens algebra $BD(2^\infty)$.
- It follows that $C^*(T_X) \rtimes \mathbb{Z}$ is SME with $BD(2^\infty)$.
- **The Basilica group**. For $X = \{x, y\}$ and $G = \langle a, b \rangle \subset \mathrm{Aut}(T_X)$, where

$$a \cdot (xw) = y(b \cdot w), \ \ a \cdot (yw) = xw,$$

$$b \cdot (xw) = x(a \cdot w), \ \ b \cdot (yw) = yw,$$

  we get an action of G on $C^*(T_X)$.
- **Question**. What are $C^*(T_X)^G, C^*(T_X) \rtimes G, K_0(C^*(T_X) \rtimes G)$?

Doplicher-Roberts algebras

- The Doplicher-Roberts algebra $\mathcal{O}_\rho$ associated to a unitary representation ρ of a compact Lie group G on $\mathcal{H} = \mathbb{C}^n$ were introduced to construct a new duality theory which strengthens the Tannaka-Krein duality.
- Consider $\rho^k : G \to U(\mathcal{H}^{\otimes k})$ the tensor power, and let

$$(\rho^m, \rho^k) = \{T : \mathcal{H}^{\otimes k} \to \mathcal{H}^{\otimes m} \mid T\rho^k = \rho^m T\}.$$

- It follows that the linear span of $\bigcup_{m,k}(\rho^m, \rho^k)$ has a natural multiplication and involution, after identifying T with $T \otimes I$.
- The Doplicher-Roberts algebra $\mathcal{O}_\rho$ is defined as the C^*-closure of the linear span of $\bigcup_{m,k}(\rho^m, \rho^k)$.
- The C^*-algebra $\mathcal{O}_\rho$ is identified with the fixed point algebra $\mathcal{O}_n^G$, where $\mathcal{O}_n$ is the Cuntz algebra.

Higher-rank DR algebras (Albandik, Meyer)

- Let G be a compact group, let $\rho_1, \rho_2, \ldots, \rho_k$ be finite dimensional representations and for $m = (m_1, \ldots, m_k) \in \mathbb{N}^k$ let $\rho^m = \rho_1^{\otimes m_1} \otimes \cdots \otimes \rho_k^{\otimes m_k}$ acting on V^m.
- Fix $\pi : G \to U(\mathcal{H})$ which contains each irreducible representation of G and construct a product system $\mathcal{E}$ over $(\mathbb{N}^k, +)$ where $\mathcal{E}_m \subseteq \mathcal{K}(\mathcal{H}, V^m \otimes \mathcal{H})$ is the space of compact intertwiners between π and $\rho^m \otimes \pi$.
- There are natural associative multiplication maps $\mathcal{E}_{m_1} \times \mathcal{E}_{m_2} \to \mathcal{E}_{m_1+m_2}$ and an $\mathcal{E}_0$-valued inner product on $\mathcal{E}_m$ given by $\langle T_1, T_2 \rangle = T_1^* T_2$ such that $\mathcal{E}_{m_1} \otimes_{\mathcal{E}_0} \mathcal{E}_{m_2} \cong \mathcal{E}_{m_1+m_2}$.
- The higher-rank Doplicher-Roberts algebra for $\rho_1, \rho_2, \ldots, \rho_k$ relative to π is the Cuntz-Pimsner algebra of the product system $(\mathcal{E}_m)_{m \in \mathbb{N}^k}$. Different choices of π give SME algebras.
- It seems unlikely that these algebras are higher-rank graph algebras.

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