Rigidity of the flag structure for a class of Cowen-Douglas operators

Dinesh Kumar Keshari, NISER, India

(Joint work with Kui Ji, Chunlan Jiang, and Gadadhar Misra)

Multivariable Operator Theory Workshop
On the occasion of Baruch Solel's 65th Birthday
Technion, Haifa: June 18-22, 2017

Motivation

- Notation:-
  $\mathscr{H}$: Separable Hilbert Space,
  $\mathscr{L}(\mathscr{H})$: all bounded linear operators on $\mathscr{H}$.

Motivation

- Notation:-
  $\mathscr{H}$: Separable Hilbert Space,
  $\mathscr{L}(\mathscr{H})$: all bounded linear operators on $\mathscr{H}$.
- Question:- Given $T_1, T_2 \in \mathscr{L}(\mathscr{H})$, when there exists a unitary operator U on $\mathscr{H}$ such that $T_2 = U^* T_1 U$.

Motivation

- Notation:-
  $\mathscr{H}$: Separable Hilbert Space,
  $\mathscr{L}(\mathscr{H})$: all bounded linear operators on $\mathscr{H}$.

- Question:- Given $T_1, T_2 \in \mathscr{L}(\mathscr{H})$, when there exists a unitary operator U on $\mathscr{H}$ such that $T_2 = U^* T_1 U$.
- In general, solution of this problem is not easy but for some special classes of operators one has affirmative answer.

- For example, a normal operator on a finite dimensional space. Eigenvalues together with their multiplicity determine the operator up to unitary equivalence.

- For example, a normal operator on a finite dimensional space. Eigenvalues together with their multiplicity determine the operator up to unitary equivalence.
- In case of infinite dimensional Hilbert space result is essentially the same.

- For example, a normal operator on a finite dimensional space. Eigenvalues together with their multiplicity determine the operator up to unitary equivalence.
- In case of infinite dimensional Hilbert space result is essentially the same.
- Set of eigenvalues of a normal operator defined on a separable Hilbert space is at most countable.

- There exist operators whose point spectrum contains an open set.

- There exist operators whose point spectrum contains an open set.
- For example, the backward shift S^* on $\ell^2(\mathbb{N})$ defined by

$$S^*(\alpha_0, \alpha_1, \alpha_2, \dots) = (\alpha_1, \alpha_2, \alpha_3, \dots)$$

for $(\alpha_0, \alpha_1, \alpha_2, \dots) \in \ell^2(\mathbb{N})$.

- There exist operators whose point spectrum contains an open set.
- For example, the backward shift S^* on $\ell^2(\mathbb{N})$ defined by

$$S^*(\alpha_0, \alpha_1, \alpha_2, \dots) = (\alpha_1, \alpha_2, \alpha_3, \dots)$$

for $(\alpha_0, \alpha_1, \alpha_2, \dots) \in \ell^2(\mathbb{N})$.
$(1, \lambda, \lambda^2, \dots) \in \ell^2(\mathbb{N})$ for $|\lambda| < 1$, we see that

$$\mathbb{D} \subseteq \sigma_p(S^*) \subseteq \sigma(S^*),$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$,

$$\sigma_p(S^*) := \{\alpha \in \mathbb{C} : S^* - \alpha I \text{ is not one-one}\}$$

and

$$\sigma(S^*) := \{\alpha \in \mathbb{C} : S^* - \alpha I \text{ is not invertible}\}.$$

- Cowen-Douglas initiated a systematic approach to study such operators.

- Cowen-Douglas initiated a systematic approach to study such operators.

Definition

Given a bounded, open and connected subset Ω of $\mathbb{C}$ and $n \in \mathbb{N}$, the Cowen-Douglas class $B_n(\Omega)$ consists of operators T satisfying the following conditions:

(1) $\Omega \subset \sigma(T) = \{w \in \mathbb{C} : T - w \text{ is not invertible}\}$;

(2) $\operatorname{ran}(T - w) = \mathscr{H}$ for w in Ω;

(3) $\operatorname{span}\{\ker(T - w) : w \in \Omega\}$ is dense in $\mathscr{H}$; and

(4) $\dim \ \ker(T - w) = n$ for w in Ω.

Example

$H^2 = \{f \in \mathscr{O}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n,\ \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$. $M_z^* \in B_1(\mathbb{D})$.

Example

$H^2 = \{f \in \mathscr{O}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$. $M_z^* \in B_1(\mathbb{D})$.

Example

$L_a^2(\mathbb{D}) = \{f \in \mathscr{O}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty\}$. $M_z^* \in B_1(\mathbb{D})$.

- There is a Hermitian holomorphic vector bundle E_T corresponding to each $T \in B_n(\Omega)$, where

$$E_T := \{(w,x) \in \Omega \times \mathscr{H} : x \in \ker(T-w)\}, \pi(w,x) = w.$$

- There is a Hermitian holomorphic vector bundle E_T corresponding to each $T \in B_n(\Omega)$, where

$$E_T := \{(w,x) \in \Omega \times \mathscr{H} : x \in \ker(T-w)\}, \pi(w,x) = w.$$

Theorem (Cowen-Douglas)

Operators T and $\widetilde{T}$ in $B_n(\Omega)$ are unitarily equivalent if and only if the corresponding Hermitian holomorphic vector bundles E_T and $E_{\widetilde{T}}$ are equivalent.

- Cowen-Douglas also find complete set of invariant consisting of curvature of $\mathscr{K}$ of E_T and a certain number of its covariant derivative.

- Cowen-Douglas also find complete set of invariant consisting of curvature of $\mathscr{K}$ of E_T and a certain number of its covariant derivative.
- But these invariant are not easy to calculate, usually, unless $n = 1$.

- Cowen-Douglas also find complete set of invariant consisting of curvature of $\mathscr{K}$ of E_T and a certain number of its covariant derivative.
- But these invariant are not easy to calculate, usually, unless $n = 1$.
- For $n = 1$, the curvature $\mathscr{K}_T$ of the bundles E_T is given by the formula

$$\mathscr{K}_T(w) \;=\; \tfrac{\partial^2}{\partial w \partial \bar{w}} \log \|\gamma(w)\|^2 d\bar{w} \wedge dw$$

for some non zero section γ of E_T.

- Cowen-Douglas also find complete set of invariant consisting of curvature of $\mathscr{K}$ of E_T and a certain number of its covariant derivative.
- But these invariant are not easy to calculate, usually, unless $n = 1$.
- For $n = 1$, the curvature $\mathscr{K}_T$ of the bundles E_T is given by the formula

$$\mathscr{K}_T(w) \quad = \quad \frac{\partial^2}{\partial w \partial \bar{w}} \log \|\gamma(w)\|^2 d\bar{w} \wedge dw$$

  for some non zero section γ of E_T.
- Operators T and $\tilde{T}$ in $B_1(\Omega)$ are equivalent if and only if the curvatures $\mathscr{K}_T(w)$ and $\mathscr{K}_{\tilde{T}}(w)$ are equal.

Theorem (Jiang-Wang)

Given $T \in B_n(\Omega)$, there exist $B_1(\Omega)$ operators $T_0, T_1, \ldots, T_{n-1}$ such that

$$T = \begin{pmatrix} T_0 & & & * \\ & T_1 & & \\ & & \ddots & \\ & 0 & & T_{n-1} \end{pmatrix}$$

with respect to some decomposition of $\mathscr{H} = \oplus_{i=1}^{n} \mathscr{H}_i$ of Hilbert space $\mathscr{H}$.

- Converse of above theorem is also true.

- Converse of above theorem is also true.
- First we will discus case when $n = 2$.

- Converse of above theorem is also true.
- First we will discus case when $n = 2$.

Definition

We let $\mathscr{F}B_2(\Omega)$ denote the set of operators of the form

$$T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$$

for some $T_o, T_1 \in B_1(\Omega)$ and non zero operator S such that $T_0 S = S T_1$.

Definition

An operator T is called homogeneous if $\sigma(T) \subseteq \bar{\mathbb{D}}$ and $\phi(T) \cong T$ for all $\phi \in$ möb.

Example

If $T \in B_2(\mathbb{D})$, irreducible and homogeneous then $T \in \mathscr{F}B_2(\mathbb{D})$.

Theorem

Every operators in $\mathscr{F}B_2(\Omega)$ are irreducible.

Theorem

Every operators in $\mathscr{F}B_2(\Omega)$ are irreducible.

- To prove above Theorem, we need to introduce some notation:
  - $\sigma_T : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$, $\sigma_T(X) = TX - XT$, where $T \in \mathscr{L}(\mathscr{H})$.

Theorem

Every operators in $\mathscr{F}B_2(\Omega)$ are irreducible.

- To prove above Theorem, we need to introduce some notation:
  - $\sigma_T : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$, $\sigma_T(X) = TX - XT$, where $T \in \mathscr{L}(\mathscr{H})$.
  - $T \in \mathscr{L}(\mathscr{H})$, is quasi-nilpotent if $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$

Theorem (Kleinecke, Shirokov)

For $P, T \in \mathscr{L}(\mathscr{H})$, if $P \in \ker \sigma_T \cap ran\, \sigma_T$ then P is quasi-nilpotent.

Lemma

For $T, \tilde{T} \in B_1(\Omega)$ and $X \in \mathscr{L}(\mathscr{H})$ such that $XT = \tilde{T}X$, then X has dense range if and only if X is non zero.

Lemma

Let $T \in B_1(\Omega)$ and $X \in \mathscr{L}(\mathscr{H})$ is quasi-nilpotent such that $XT = TX$ then $X = 0$.

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$
- $PT = TP \Rightarrow P_{21}S \in \ker \sigma_{T_1} \cap \operatorname{ran} \sigma_{T_1}$

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$
- $PT = TP \Rightarrow P_{21}S \in \ker \sigma_{T_1} \cap \operatorname{ran} \sigma_{T_1}$
- By previous lemmas, $P_{21}S = 0$

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$
- $PT = TP \Rightarrow P_{21}S \in \ker \sigma_{T_1} \cap \operatorname{ran} \sigma_{T_1}$
- By previous lemmas, $P_{21}S = 0$
- S has dense range $\Rightarrow P_{21} = 0$

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$
- $PT = TP \Rightarrow P_{21}S \in \ker \sigma_{T_1} \cap \operatorname{ran} \sigma_{T_1}$
- By previous lemmas, $P_{21}S = 0$
- S has dense range $\Rightarrow P_{21} = 0$
- P is self adjoint $\Rightarrow P_{12} = 0$

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$
- $PT = TP \Rightarrow P_{21}S \in \ker \sigma_{T_1} \cap \operatorname{ran} \sigma_{T_1}$
- By previous lemmas, $P_{21}S = 0$
- S has dense range $\Rightarrow P_{21} = 0$
- P is self adjoint $\Rightarrow P_{12} = 0$
- P_{11} and P_{22} are projections such that $P_{11}T_0 = T_0P_{11}$ and $P_{22}T_1 = T_1P_{22}$

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$
- $PT = TP \Rightarrow P_{21}S \in \ker \sigma_{T_1} \cap \operatorname{ran} \sigma_{T_1}$
- By previous lemmas, $P_{21}S = 0$
- S has dense range $\Rightarrow P_{21} = 0$
- P is self adjoint $\Rightarrow P_{12} = 0$
- P_{11} and P_{22} are projections such that $P_{11}T_0 = T_0P_{11}$ and $P_{22}T_1 = T_1P_{22}$
- $P_{11} = I$ or 0 and $P_{22} = I$ or 0

Proof.

- Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ be a projection such that $PT = TP$
- $PT = TP \Rightarrow P_{21}S \in \ker \sigma_{T_1} \cap \operatorname{ran} \sigma_{T_1}$
- By previous lemmas, $P_{21}S = 0$
- S has dense range $\Rightarrow P_{21} = 0$
- P is self adjoint $\Rightarrow P_{12} = 0$
- P_{11} and P_{22} are projections such that $P_{11}T_0 = T_0P_{11}$ and $P_{22}T_1 = T_1P_{22}$
- $P_{11} = I$ or 0 and $P_{22} = I$ or 0
- $P_{11}S = SP_{22} \Rightarrow P_{11} = P_{22} = 0$ or $P_{11} = P_{22} = I$

$\square$

Any intertwining unitary between two operators in $\mathscr{F}B_2(\Omega)$ is diagonal, in other words,

Theorem

Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ *be a unitary operator such that*

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

then $U_{12} = U_{21} = 0$.

Corollary

$T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, $\tilde{T} = \begin{pmatrix} T_0 & \tilde{S} \\ 0 & T_1 \end{pmatrix}$; $T_0, T_1 \in B_1(\Omega)$, $T_0 S = S T_1$ and $T_0 \tilde{S} = \tilde{S} T_1$.

$$T \cong \tilde{T} \Leftrightarrow \tilde{S} = e^{i\theta} S, \quad \textit{for some } \theta \in \mathbb{R}.$$

Corollary

$T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, $\tilde{T} = \begin{pmatrix} T_0 & \tilde{S} \\ 0 & T_1 \end{pmatrix}$; $T_0, T_1 \in B_1(\Omega)$, $T_0 S = S T_1$ and $T_0 \tilde{S} = \tilde{S} T_1$.

$$T \cong \tilde{T} \Leftrightarrow \tilde{S} = e^{i\theta} S, \quad \text{for some } \theta \in \mathbb{R}.$$

Corollary

Let $T_i : \mathscr{H}_i \to \mathscr{H}_i$, $i = 0, 1$ be bounded linear operators and $T_1, T_0 \in \mathscr{B}_1(\Omega)$. Let S be a bounded linear operators such that $ST_1 = T_0 S$. Let μ be positive real number. Set, $T_\mu = \begin{pmatrix} T_0 & \mu S \\ 0 & T_1 \end{pmatrix}$. T_μ is unitarily equivalent to $T_{\tilde{\mu}}$ if and only if $\mu = \tilde{\mu}$.

Theorem

Let $T, \tilde{T} \in \mathscr{F}B_2(\Omega)$, $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$. Let t_1 and $\tilde{t}_1$ be non zero section of E_{T_1} and $E_{\tilde{T}_1}$ respectively.

$$T \cong \tilde{T} \Leftrightarrow \mathscr{K}_{T_0} = \mathscr{K}_{\tilde{T}_0}, \ \frac{\|S(t_1)(w)\|}{\|t_1(w)\|} = \frac{\|\tilde{S}(\tilde{t}_1)(w)\|}{\|\tilde{t}_1(w)\|}, \ \text{for all } w \in \Omega.$$

Equivalently,

$$T \cong \tilde{T} \Leftrightarrow \mathscr{K}_{T_0} = \mathscr{K}_{\tilde{T}_0}, \text{ and } \theta_{12} = \tilde{\theta}_{12},$$

where θ_{12} is second fundamental form of E_{T_0} in E_T.

Proof.

$\Rightarrow$

- $\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$
- $U_1(S(t_1)) = \phi_1(\tilde{S}(\tilde{t}_1))$ and $U_2 t_1 = \phi_2 \tilde{t}_1$
- $U_1 S = \tilde{S} U_2 \Rightarrow \phi_1 = \phi_2$
- $\mathscr{K}_{T_0} = \mathscr{K}_{\tilde{T}_0}$ and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$

Proof.

$\Rightarrow$

- $\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$
- $U_1(S(t_1)) = \phi_1(\tilde{S}(\tilde{t}_1))$ and $U_2 t_1 = \phi_2 \tilde{t}_1$
- $U_1 S = \tilde{S} U_2 \Rightarrow \phi_1 = \phi_2$
- $\mathscr{K}_{T_0} = \mathscr{K}_{\tilde{T}_0}$ and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$

$\Leftarrow$

- Set, $\gamma_0(w) := S(t_1(w))$, $\tilde{\gamma}_0(w) := \tilde{S}(\tilde{t}_1(w))$,
  $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$ and $\tilde{\gamma}_1(w) := \frac{\partial}{\partial w}\tilde{\gamma}_0(w) - \tilde{t}_1(w)$
- $\mathscr{K}_{T_0} = \mathscr{K}_{\tilde{T}_0} \Rightarrow \|\gamma_0\|^2 = |\phi|^2 \|\tilde{\gamma}_0\|^2$
- $\frac{\|\gamma_0\|^2}{\|t_1\|^2} = \frac{\|\tilde{\gamma}_0\|^2}{\|\tilde{t}_1\|^2} \Rightarrow \|t_1\|^2 = |\phi|^2 \|\tilde{t}_1\|^2$
- $\Psi : E_T \to E_{\tilde{T}}$, $\Psi(\gamma_0) = \phi\tilde{\gamma}_0$ and $\Psi(\gamma_1) = \phi'\tilde{\gamma}_0 + \phi\tilde{\gamma}_1$
- $\langle \Psi(\gamma_i), \Psi(\gamma_j) \rangle = \langle \gamma_i, \gamma_j \rangle$, for $0 \le i, j \le 1$.

$\square$

Theorem

$T \in \mathscr{F}B_2(\mathbb{D})$, $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$.

$$T \text{ is a homogeneous operator} \Leftrightarrow$$

(1) T_0 *and* T_1 *are homogeneous;*

(2) $\mathscr{K}_{T_0} = \mathscr{K}_{T_1} + \mathscr{K}_{B^*}$, *B is Bergman shift;*

(3) $S(t_1(w)) = \alpha\gamma_0(w)$ *where* $\|t_1(w)\|^2 = \dfrac{1}{(1-|w|^2)^\mu}$ *and*

$$\|\gamma_0(w)\|^2 = \frac{1}{(1-|w|^2)^\lambda}, \lambda, \mu \in \mathbb{R}_+.$$

Definition

We let $\mathscr{F}B_n(\Omega)$ be the set of all operators T defined on complex separable Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \cdots \oplus \mathscr{H}_{n-1}$ which is of the form

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix}$$

where $T_i : \mathscr{H}_i \to \mathscr{H}_i$, defined on the complex separable Hilbert space $\mathscr{H}_i$, $0 \le i \le n-1$, is assumed to be in $B_1(\Omega)$ and $S_{i,i+1} : \mathscr{H}_{i+1} \to \mathscr{H}_i$, is assumed to be a non-zero intertwining operator, namely, $T_i S_{i,i+1} = S_{i,i+1} T_{i+1}, 0 \le i \le n-2$.

Theorem

Every operators in $\mathscr{F}B_n(\Omega)$ are irreducible.

Theorem

Every operators in $\mathscr{F}B_n(\Omega)$ *are irreducible.*

Theorem

$T, \tilde{T} \in \mathscr{F}B_n(\Omega)$.
$T \cong \tilde{T} \Leftrightarrow$ *there exist unitary operators* $U_i : \mathscr{H}_i \to \tilde{\mathscr{H}}$, $i = 0, 1, \cdots n-1$ *such that* $U_i \tilde{T}_i U_i^* = T_i$ *and* $U_i S_{i,j} = \tilde{S}_{i,j} U_j$, *for any* $i < j$.

Theorem

Suppose T is an operator in $\mathscr{F}B_n(\Omega)$ and that t_{n-1} is a non-vanishing section of $E_{T_{n-1}}$. Then

- *the curvature $\mathscr{K}_{T_0}$,*
- $\frac{\|t_{i-1}\|}{\|t_i\|}$*, where $t_{i-1} = S_{i-1,i}(t_i), 1 \leq i \leq n-1$;*
- $\frac{\langle S_{k,l}(t_l), t_k\rangle}{\|t_k\|^2}, 0 \leq k < l \leq n-2$ *with $i - j \geq 2$.*

are a complete set of unitary invariants for the operator T.

Corollary

$$T = \begin{pmatrix} T_0 & S_{0,1} & 0 & \cdots & 0 \\ 0 & T_1 & S_{1,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & 0 & 0 & 0 & T_{n-1} \end{pmatrix}_{n\times n},$$

$$\tilde{T} = \begin{pmatrix} T_0 & \tilde{S}_{0,1} & 0 & \cdots & 0 \\ 0 & T_1 & \tilde{S}_{1,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & T_{n-2} & \tilde{S}_{n-2,n-1} \\ 0 & 0 & 0 & 0 & T_{n-1} \end{pmatrix}_{n\times n};$$

$$T \cong \tilde{T} \Leftrightarrow \tilde{S}_{j,j+1} = e^{i\theta_j} S_{j,j+1}, \theta_j \in \mathbb{R}, 0 \leq j \leq n-2.$$

Corollary

$$T_\mu = \begin{pmatrix} T_0 & \mu_0 S_{0,1} & 0 & \cdots & 0 \\ 0 & T_1 & \mu_1 S_{1,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & T_{n-2} & \mu_{n-2} S_{n-2,n-1} \\ 0 & 0 & 0 & 0 & T_{n-1} \end{pmatrix},$$

where $\mu_i > 0, 0 \leq i \leq n-2$, $\mu = (\mu_0, \ldots, \mu_{n-2})$. T_μ is unitarily equivalent to $T_{\tilde{\mu}}$ if and only if $\mu = \tilde{\mu}$.

Thank You and Happy Birthday to Prof. Baruch Solel