Schatten-class perturbations of Toeplitz operators

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Multivariable Operator Theory, 2017

Joint work with
Michael Didas and Jörg Eschmeier
(both Saarland University)
Definition

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ with the canonical probability measure $m$. The *Hardy space with respect to $m$* will be denoted by

$$H^2(m) = \left\{ f \in L^2(m) \ ; \ \hat{f}(n) = 0 \text{ for all } n < 0 \right\} \subset L^2(m).$$
The classical problem

Formulation of the problem

Main result

Analytic Toeplitz Operators

Definition

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Define

$$H^\infty(m) = L^\infty(m) \cap H^2(m) \subset L^\infty(m).$$
Definition

Let \( \mathbb{T} \) be the unit circle in \( \mathbb{C} \) with the canonical probability measure \( m \). The *Hardy space with respect to \( m \) will be denoted by*

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H^\infty(m) = L^\infty(m) \cap H^2(m) \subset L^\infty(m)
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and we call

\[
l_m = \{ f \in H^\infty(m) \ ; \ |f| = 1 \text{ } m\text{-a.e.} \}
\]

the set of *inner functions with respect to \( m \).*
Definition

Let $f \in L^\infty(m)$. We call

$$T_f : H^2(m) \rightarrow H^2(m), \quad g \mapsto P_{H^2(m)}(fg),$$

where $P_{H^2(m)} : L^2(m) \rightarrow H^2(m)$ is the orthogonal projection onto $H^2(m)$, the **Toeplitz operator with symbol** $f$. 

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Schatten-class perturbations of Toeplitz operators
Theorem (Brown-Halmos condition I, 1964)

An operator $X \in B(H^2(m))$ is a Toeplitz operator (i.e. there exists a $f \in L^\infty(m)$ such that $X = T_f$) if and only if

$$T_z^* X T_z - X = 0,$$

where $z \in L^\infty(m)$ is the identity map.
### Theorem (Brown-Halmos condition I, 1964)

An operator $X \in B(H^2(m))$ is a Toeplitz operator (i.e. there exists a $f \in L^\infty(m)$ such that $X = T_f$) if and only if

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### Theorem (Brown-Halmos condition II)

An operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if

$$T_u^*XT_u - X = 0$$

for all $u \in I_m$. 

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**Theorem (Gu, 2004)**

An operator $X \in B(H^2(m))$ is a finite rank Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $F \in F(H^2(m))$ such that $X = T_f + F$) if and only if

$$T_u^* X T_u - X \in F(H^2(m))$$

for all $u \in l_m$. 

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Theorem (Xia, 2009)

An operator $X \in B(H^2(m))$ is a compact Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $K \in K(H^2(m))$ such that $X = T_f + K$) if and only if

$$T_u^*XT_u - X \in K(H^2(m))$$

for all $u \in l_m$. 
Definition

Let $p \in [1, \infty)$ and let $H$ be a Hilbert space. An operator $S \in B(H)$ is a Schatten-$p$-class operator if

$$\|S\|_p^p = \text{tr}(|S|^p) < \infty.$$
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Let $p \in [1, \infty)$ and let $H$ be a Hilbert space. An operator $S \in B(H)$ is a \textit{Schatten-$p$-class} operator if

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Furthermore, we set

$$S_p(H) = \left\{ S \in B(H) \; ; \; \|S\|_p < \infty \right\}$$

equipped with $\|\cdot\|_p$. 
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equipped with $\|\cdot\|_p$ and

$$S_0(H) = F(H)$$

as well as

$$S_\infty(H) = K(H)$$

both equipped with the operator norm.
Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with the canonical probability measure $m \in M_1^+(\mathbb{T})$.

**Theorem (Gu, Xia, Didas-Eschmeier-S.)**

Let $p \in \{0\} \cup [1, \infty]$. An operator $X \in B(H^2(m))$ is a $S_p(H^2(m))$ Toeplitz perturbation (i.e. there exist a $f \in L^\infty(m)$ and $S \in S_p(H^2(m))$ such that $X = T_f + S$) if and only if

$$T_u^* X T_u - X \in S_p(H^2(m))$$

for all $u \in l_m$. 

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**Schatten-class perturbations of Toeplitz operators**
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$$A(\mathbb{D}) = \{ f \in C(\overline{\mathbb{D}}) ; \ f|_\mathbb{D} \in \mathcal{O}(\mathbb{D}) \}$$

the *disc algebra*
Definition

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $\mathcal{O}(\mathbb{D})$ be the set of all scalar-valued analytic functions on $\mathbb{D}$. We call

$$A(\mathbb{D}) = \{ f \in C(\overline{\mathbb{D}}) \mid f|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}) \}$$

the *disc algebra* and denote by $\partial A(\mathbb{D})$ the *Shilov boundary* of $A(\mathbb{D})$ (i.e. the smallest closed subset of $\overline{\mathbb{D}}$ such that

$$\sup_{z \in \overline{\mathbb{D}}} |f(z)| = \sup_{z \in \partial A(\mathbb{D})} |f(z)| \text{ for all } f \in A(\mathbb{D}).$$
### Definition

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$$\sup_{z \in \mathbb{D}} |f(z)| = \sup_{z \in \partial A(\mathbb{D})} |f(z)| \quad \text{for all } f \in A(\mathbb{D}).$$

### Proposition

*We have*

$$\partial A(\mathbb{D}) = \partial \mathbb{D} = \mathbb{T}.$$
Definition

Let $D \subset \mathbb{C}^n$ be a bounded domain. We denote by

(i) $A(D) = \{ f \in C(\overline{D}) ; f|_D \in \mathcal{O}(D) \} \subset C(\overline{D})$ the *domain algebra* of $D$, 

(ii) $S = \partial A(D)$ the *Shilov boundary* of $A(D)$ (i.e. the smallest closed subset of $\overline{D}$ such that $\sup_{z \in D} |f(z)| = \sup_{z \in S} |f(z)|$ for all $f \in A(D)$).
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Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex or a bounded symmetric and circled domain. We denote by $\mu \in M_1^+(S)$ the canonical probability measure on $S$. 
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**Examples**

(i) $D = B_n$: $S = \partial B_n$ and $\mu = \sigma$.
(ii) $D = D^n$: $S = T^n$ and $\mu = \otimes_n m$. 
We have

\[ H^2(m) = A(\mathbb{D})|_T T \| \cdot \|_{L^2(m)} \quad \text{and} \quad H^\infty(m) = A(\mathbb{D})|_T T w^*. \]
We have

\[ H^2(m) = \overline{A(D)}|_T \mathcal{T}_{\| \cdot \|} L^2(m) \quad \text{and} \quad H^\infty(m) = \overline{A(D)}|_T \mathcal{T}_{w^*}. \]

**Definition**

We define

\[ H^2(\mu) = \overline{A(D)}|_S \mathcal{T}_{\| \cdot \|} L^2(\mu) \subset L^2(\mu) \]

and

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We have

\[ H^2(m) = \overline{A(D)|T\|\cdot\|L^2(m)} \quad \text{and} \quad H^\infty(m) = \overline{A(D)|T^w}. \]

**Definition**

We define

\[ H^2(\mu) = \overline{A(D)|S^{T\|\cdot\|L^2(\mu)}} \subset L^2(\mu) \]

and

\[ H^\infty(\mu) = \overline{A(D)|S^{T^w^*}} \subset L^\infty(\mu). \]

Furthermore, we denote by

\[ I_\mu = \{ f \in H^\infty(\mu) ; \ |f| = 1 \ \mu\text{-a.e.} \} \]

the set of inner functions with respect to \( \mu \).
Definition

Let \( f \in L^\infty(\mu) \). We call

\[ T_f : H^2(\mu) \to H^2(\mu), \; g \mapsto P_{H^2(\mu)}(fg), \]

where \( P_{H^2(\mu)} : L^2(\mu) \to H^2(\mu) \) is the orthogonal projection onto \( H^2(\mu) \), the Toeplitz operator with symbol \( f \).
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**Theorem (Didas-Eschmeier-Everard, 2011)**

An operator \( X \in B(H^2(\mu)) \) is a Toeplitz operator if and only if

\[
T_u^* XT_u - X = 0
\]

for all \( u \in I_\mu \).
**Definition**

We denote by $H^\infty(\mathbb{D}) \subset \mathcal{O}(\mathbb{D})$ the set of all bounded analytic functions on $\mathbb{D}$. 
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Theorem

The map

$$r_m : H^\infty(\mathbb{D}) \to H^\infty(m), \, \theta \mapsto \tau_{\mathcal{W}^*} \lim_{r \to 1} [\theta(r \cdot)] =: \theta^*$$

is an isometric algebra isomorphism and weak* homeomorphism with $r_m(\theta|_\mathbb{D}) = [\theta|_\mathcal{T}]$ for all $\theta \in A(\mathbb{D})$. 
<table>
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<tr>
<th>Definition</th>
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We denote by $H^\infty(D) \subset \mathcal{O}(D)$ the set of all bounded analytic functions on $D$.

Theorem

There exists a map

$$r_\mu : H^\infty(D) \to H^\infty(\mu), \quad \theta \mapsto r_\mu(\theta) =: \theta^*,$$

which is an isometric algebra isomorphism and weak* homeomorphism with $r_\mu(\theta|_D) = [\theta|_S]$ for all $\theta \in A(D)$. 
Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex or bounded symmetric and circled domain and $\mu \in M_1^+(S)$ be the probability measure on the Shilov boundary $S = \partial A(D)$ obtained before.

**Theorem (Didas-Eschmeier-S., 2017)**

Let $p \in [1, \infty)$. An operator $X \in B(H^2(\mu))$ is a $S_p(H^2(\mu))$ Toeplitz perturbation (i.e. there exist a $f \in L^\infty(\mu)$ and $S \in S_p(H^2(\mu))$ such that $X = T_f + S$) if and only if

$$T_u^*XT_u - X \in S_p(H^2(\mu))$$

for all $u \in l_\mu$. 

**Proposition**

Let \((\alpha_k)_{k \in \mathbb{N}}\) be a sequence in \(H^\infty(\mu)\) with

\[
    \tau_{W^*} \lim_{k \to \infty} \alpha_k = \alpha \in [0, 1)
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Proposition

Let \((\alpha_k)_{k \in \mathbb{N}}\) be a sequence in \(H^\infty(\mu)\) with

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and \(X \in B(H^2(\mu))\) an operator such that

\[
Y = \tau_{\text{WOT}} \lim_{k \to \infty} T^*_\alpha_k X T_{\alpha_k} \in B(H^2(\mu))
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exists.
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Let \((\alpha_k)_{k \in \mathbb{N}}\) be a sequence in \(H^\infty(\mu)\) with

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exists. If \(T^*_u XT_u - X \in S_\infty(H^2(\mu))\) for all \(u \in I_\mu\), then there exists a function \(f \in L^\infty(\mu)\) such that

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X = T_f + \frac{1}{1 - \alpha^2}(X - Y).
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Proposition

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Proposition (Hiai, 1997)

The map

\[ \| \cdot \|_p : (B(H^2(\mu)), \tau_{WOT}) \rightarrow [0, \infty], \ S \mapsto \| S \|_p \]

is lower semi-continuous.
Proposition (Hiai, 1997)

The map

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is lower semi-continuous.

$$\left\| \tau_{WOT} \lim_{k \to \infty} X - T^{\ast}_{\alpha_k} X T_{\alpha_k} \right\|_p \leq \liminf_{k \to \infty} \left\| X - T^{\ast}_{\alpha_k} X T_{\alpha_k} \right\|_p$$
We denote by

\[ l_D = \{ \theta \in H^\infty(D) ; \theta^* \in l_{\mu} \} \]

the set of inner functions with respect to \( D \) and \( \mu \).
We denote by
\[ I_D = \{ \theta \in H^\infty(D) ; \theta^* \in l_\mu \} \]
the set of inner functions with respect to $D$ and $\mu$.

**Proposition (Aleksandrov, 1984)**

*Let* $\alpha \in [0, 1)$. *Then there exists a sequence* $(\alpha_k)_{k \in \mathbb{N}}$ *in* $I_D$ *such that*

\[ \tau_{w^*} - \lim_{k \to \infty} \alpha_k^* = \alpha. \]
Proposition (Xia, 2009)

Let $p \in [1, \infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

$$T_u^*XT_u - X \in S_p(H^2(\mu))$$

for all $u \in l_\mu$. 
Proposition (Xia, 2009)

Let $p \in [1, \infty]$. Suppose that $X \in B(H^2(\mu))$ is an operator such that

$$T_u^* X T_u - X \in S_p(H^2(\mu))$$

for all $u \in l_\mu$. Then, for all $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\|T_{\theta}^* X T_{\theta}^* - X\|_p \leq \varepsilon$$

for all $\theta \in l_D$ with $\left|\int_S 1 - \theta^* \, d\mu\right| \leq \delta$. 
Proof.

There exists $0 < \delta < 1$ such that $\| T_{\theta^*} X T_{\theta^*} - X \|_p \leq 1$ for all $\theta \in l_D$ with $\left| \int_S 1 - \theta^* \, d\mu \right| \leq \delta$. Set $\alpha = 1 - \delta/2$. 
Proof.

There exists $0 < \delta < 1$ such that $\| T_\theta^* X T_\theta^* - X \|_p \leq 1$ for all $\theta \in I_D$ with $| \int_S 1 - \theta^* \, d\mu | \leq \delta$. Set $\alpha = 1 - \delta/2$.

$\implies$ There exists $(\alpha_k)_{k \in \mathbb{N}}$ in $I_D$ with $\tau_{w^*}-\lim_{k \to \infty} \alpha_k^* = \alpha$. 

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Proof.

There exists $0 < \delta < 1$ such that $\| T_{\theta^*} X T_{\theta^*} - X \|_p \leq 1$ for all $\theta \in I_D$ with $\left| \int_S 1 - \theta^* \, d\mu \right| \leq \delta$. Set $\alpha = 1 - \delta/2$.

$\implies$ There exists $(\alpha_k)_{k \in \mathbb{N}}$ in $I_D$ with $\tau_{w^*} - \lim_{k \to \infty} \alpha_k^* = \alpha$.

By passing to a subsequence we can achieve that $\left| \int_D 1 - \alpha_k^* \, d\mu \right| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{WOT} - \lim_{k \to \infty} T_{\alpha_k^*} X T_{\alpha_k^*} \in B(H^2(\mu))$$

exists.
Proof.

There exists $0 < \delta < 1$ such that $\| T_{\theta^*}^* X T_{\theta^*} - X \|_p \leq 1$ for all $\theta \in \mathcal{D}$ with $\left| \int_S 1 - \theta^* \, d\mu \right| \leq \delta$. Set $\alpha = 1 - \delta/2$.

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exists. Hence

$$\left\| \tau_{\text{WOT}} \lim_{k \to \infty} X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \leq \liminf_{k \to \infty} \left\| X - T_{\alpha_k^*}^* X T_{\alpha_k^*} \right\|_p \leq 1.$$
Proof.

There exists $0 < \delta < 1$ such that $\| T_{\theta^*} X T_{\theta^*} - X \|_p \leq 1$ for all $\theta \in \mathcal{I}_D$ with $\left| \int_S 1 - \theta^* \, d\mu \right| \leq \delta$. Set $\alpha = 1 - \delta/2$.

$\implies$ There exists $(\alpha_k)_{k \in \mathbb{N}}$ in $\mathcal{I}_D$ with $\tau_{w^*}-\lim_{k \to \infty} \alpha_k^* = \alpha$.

By passing to a subsequence we can achieve that $\left| \int_D 1 - \alpha_k^* \, d\mu \right| \leq \delta$ for all $k \in \mathbb{N}$ and that at the same time the limit

$$Y = \tau_{\text{WOT}}-\lim_{k \to \infty} T_{\alpha_k^*} X T_{\alpha_k^*} \in B(H^2(\mu))$$

exists. Hence

$$\left\| \tau_{\text{WOT}}-\lim_{k \to \infty} X - T_{\alpha_k^*} X T_{\alpha_k^*} \right\|_p \leq \liminf_{k \to \infty} \left\| X - T_{\alpha_k^*} X T_{\alpha_k^*} \right\|_p \leq 1.$$

$\implies$ $X - Y \in S_p(H^2(\mu))$. 

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(ii) The measure \(\mu\) is a faithful Henkin measure.
Remark

The following ingredients are essential for the proof.

(i) The triple \((A(D)|_S, S, \mu)\) is regular (in the sense of Aleksandrov).

(ii) The measure \(\mu\) is a faithful Henkin measure.

Question

What about \(p = 0, \infty\)?
The classical problem

Formulation of the problem

Main result

Analytic Toeplitz Operators

**Theorem**

An operator $X \in B(H^2(m))$ is an analytic Toeplitz operator (i.e. there exists a $f \in H^\infty(m)$ such that $X = T_f$) if and only if

$$[X, T_g] = XT_g - T_gX = 0.$$ 

for all $g \in H^\infty(m)$. 

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for all $g \in H^\infty(m)$.

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**Theorem (Davidson, 1977)**

An operator $X \in B(H^2(m))$ has the form $X = T_f + K$ with $f \in H^\infty(m) + C(\mathbb{T})$ and $K \in K(H^2(m))$ if and only if

$$XT_g - T_g X \in K(H^2(m)).$$

for all $g \in H^\infty(m)$. 

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Schatten-class perturbations of Toeplitz operators
Theorem (Hartman, 1958)

We have

\[ H^\infty(m) + C(\mathbb{T}) = \{ f \in L^\infty(m) ; H_f \text{ is compact} \} \]

with \( H_f = (1 - P_{H^2(m)}) M_f \big|_{H^2(m)} \) for \( f \in L^\infty(m) \).
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Theorem (Hartman, 1958)

We have

$$H^\infty(m) + C(\mathbb{T}) = \{ f \in L^\infty(m) \ ; \ H_f \text{ is compact} \}$$

with $H_f = (1 - P_{H^2(m)})M_f|_{H^2(m)}$ for $f \in L^\infty(m)$.

Theorem (Davidson, Hartman)

An operator $X \in B(H^2(m))$ has the form $X = T_f + K$ with $f \in L^\infty(m)$ such that $H_f$ is compact and $K \in K(H^2(m))$ if and only if

$$XT_g - T_gX \in K(H^2(m)).$$

for all $g \in H^\infty(m)$. 
Let $n \geq 1$ and

$$D = \mathbb{B}_n.$$
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$$D = \mathbb{B}_n.$$ 

**Theorem (Ding-Sun, 1997)**

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + K$ for some $f \in L^\infty(\mu)$ such that $H_f$ is compact and $K \in K(H^2(\mu))$ if and only if

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for all $g \in H^\infty(\mu)$. 

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Let \( n \geq 1 \) and

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D = \mathbb{B}_n \quad \text{or} \quad D = \mathbb{D}^n.
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**Theorem (Guo-Wang, 2006)**

An operator $X \in B(H^2(\mu))$ has the form $X = T_f + F$ for some $f \in L^\infty(\mu)$ such that $H_f$ is of finite rank and $F \in F(H^2(\mu))$ if and only if

$$XT_g - T_gX \in F(H^2(\mu))$$

for all $g \in H^\infty(\mu)$. 
Let \( n \geq 1 \) and

\[
D = \mathbb{B}_n \quad \text{or} \quad D = \mathbb{D}^n.
\]

**Theorem (Didas-Eschmeier-S., 2017)**

An operator \( X \in B(H^2(\mu)) \) has the form \( X = T_f + S \) for some \( f \in L^\infty(\mu) \) such that \( H_f \) is in the Schatten-\( p \)-class and \( S \in S_p(H^2(\mu)) \) if and only if

\[
XT_g - T_gX \in S_p(H^2(\mu))
\]

for all \( g \in H^\infty(\mu) \).