Crossed Products of Operator Algebras: applications of Takai duality.

Elias Katsoulis & Chris Ramsey

- $\mathcal{A}$: an approximately unital operator algebra

- $\mathcal{G}$: a locally compact Hausdorff group

- $\alpha : \mathcal{G} \to \mathrm{Aut}\,\mathcal{G}$, a continuous group representation via completely isometric automorphisms

Also if $\mathcal{A}$ is an approximately unital operator algebra, then

- $\mathrm{C}^*_{\mathrm{max}}(\mathcal{A})$: the maximal C*-cover of $\mathcal{A}$

- $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$: the C*-envelope of $\mathcal{A}$

DEFINITION. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system and let $(\mathcal{C}, j)$ be a C*-cover of $\mathcal{A}$. Then $(\mathcal{C}, j)$ is said to be α-admissible, if there exists a continuous group representation $\dot{\alpha} : \mathcal{G} \to$ Aut$(\mathcal{C})$ which extends the representation

$$\mathcal{G} \ni s \mapsto j \circ \alpha_s \circ j^{-1} \in \mathrm{Aut}(j(\mathcal{A})). \qquad (1)$$

Since $\dot{\alpha}$ is uniquely determined by its action on $j(\mathcal{A})$, both α and its extension $\dot{\alpha}$ will be denoted by the symbol.

DEFINITION. (Relative Crossed Product) Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system and let $(\mathcal{C}, j)$ be an α-admissible C*-cover for $\mathcal{A}$. Then,

$$\mathcal{A} \rtimes_{\mathcal{C},j,\alpha} \mathcal{G} \subseteq \mathcal{C} \rtimes_{\alpha} \mathcal{G}$$

and

$$\mathcal{A} \rtimes^{r}_{\mathcal{C},j,\alpha} \mathcal{G} \subseteq \mathcal{C} \rtimes^{r}_{\alpha} \mathcal{G}$$

will denote the subalgebras of the crossed product C*-algebras $\mathcal{C} \rtimes_{\alpha} \mathcal{G}$ and $\mathcal{C} \rtimes^{r}_{\alpha} \mathcal{G}$ respectively, which are generated by $C_c\Big(\mathcal{G}, j(\mathcal{A})\Big) \subseteq C_c\Big(\mathcal{G}, \mathcal{C}\Big)$.

DEFINITION. If $(\mathcal{A}, \mathcal{G}, \alpha)$ is a dynamical system then

$$\mathcal{A} \rtimes_\alpha \mathcal{G} \equiv \mathcal{A} \rtimes_{\mathrm{C}^*_{\max}(\mathcal{A}),\alpha} \mathcal{G}$$

PROPOSITION. Let $(\mathcal{A}, \mathcal{G}, \phi)$ be a dynamical system and let

$$\phi : \mathcal{A} \rtimes_{\mathrm{C}^*_{\mathrm{max}}(\mathcal{A}),\alpha} \mathcal{G} \longrightarrow B(\mathcal{H})$$

be a non-degenerate completely contractive representation. Then there exists a non-degenerate covariant representation $(\pi, u, \mathcal{H})$ of $(\mathcal{A}, \mathcal{G}, \phi)$ so that $\phi = \pi \rtimes u$.

In the case where $\mathcal{A}$ is a C*-algebra then $\mathcal{A} \rtimes_\alpha \mathcal{G}$ is nothing else but the full crossed product C*-algebra of $(\mathcal{A}, \mathcal{G}, \alpha)$. In the general case of an operator algebra, one might be tempted to say that $\mathcal{A} \rtimes_\alpha \mathcal{G} \simeq \mathcal{A} \rtimes_{\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}),\alpha} \mathcal{G}$. This is not so clear. The identification $\mathcal{A} \rtimes_\alpha \mathcal{G} \simeq \mathcal{A} \rtimes_{\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}),\alpha} \mathcal{G}$ is a major open problem, which is resolved however in the case where $\mathcal{G}$ is amenable.

In the case where $\mathcal{G}$ is amenable, all relative full crossed products coincide as the next result shows. Its proof requires an essential use of the theory of maximal dilations.

THEOREM. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system with $\mathcal{G}$ amenable and let $(\mathcal{C}, j)$ be an α-admissible C*-cover for $\mathcal{A}$. Then

$$\mathcal{A} \rtimes_{\alpha} \mathcal{G} \simeq \mathcal{A} \rtimes_{\mathcal{C}, j, \alpha} \mathcal{G} \simeq \mathcal{A} \rtimes^{r}_{\mathcal{C}, j, \alpha} \mathcal{G}$$

via a complete isometry that maps generators to generators.

In particular the proof establishes the fact that all relative *reduced* crossed products coincide!

An operator algebra $\mathcal{A}$ is said to be Dirichlet if

$$\overline{\mathcal{A}+\mathcal{A}^*} = \mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$$

THEOREM. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system with $\mathcal{A}$ Dirichlet and let $(\mathcal{C}, j)$ be an α-admissible C*-cover for $\mathcal{A}$. Then

$$\mathcal{A} \rtimes_{\alpha} \mathcal{G} \simeq \mathcal{A} \rtimes_{\mathcal{C}, j, \alpha} \mathcal{G}$$

via a complete isometry that maps generators to generators.

THEOREM. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system. Then

$$\mathrm{C}^*_{\mathrm{max}}\Big(\mathcal{A} \rtimes_\alpha \mathcal{G}\Big) \simeq \mathrm{C}^*_{\mathrm{max}}(\mathcal{A}) \rtimes_\alpha \mathcal{G}.$$

One of the central problems of our theory is whether or not the identity

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{A} \rtimes_\alpha \mathcal{G}) = \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \rtimes_\alpha \mathcal{G}. \tag{2}$$

is valid. Fortunately in the case where $\mathcal{G}$ is an abelian group we show that the above identity is indeed valid.

THEOREM. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a unital dynamical system and assume that $\mathcal{G}$ is an abelian locally compact group. Then

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{A} \rtimes_\alpha \mathcal{G}) \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \rtimes_\alpha \mathcal{G}.$$

THEOREM (Katsoulis 2016). Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a unital dynamical system and assume that $\mathcal{G}$ is a discrete group. Then

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{A} \rtimes^r_\alpha \mathcal{G}) \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \rtimes^r_\alpha \mathcal{G}.$$

In particular, if $\mathcal{G}$ is amenable,

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{A} \rtimes_\alpha \mathcal{G}) \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \rtimes_\alpha \mathcal{G}.$$

Here is the promised version of Takai duality for *arbitrary* operator algebras. We will make shortly an important use of that duality in our investigation for the semisimplicity of crossed products.

Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system with $\mathcal{G}$ an abelian locally compact group. Let $\hat{\mathcal{G}}$ be the Pontryagin dual of $\mathcal{G}$. The dual action $\hat{\alpha}$ is defined on $C_c(\mathcal{G}, \mathcal{A})$ by $\hat{\alpha}_\gamma(f)(s) = \overline{\gamma(s)} f(s)$, $f \in C_c(\mathcal{G}, \mathcal{A})$, $\gamma \in \hat{\mathcal{G}}$.

THEOREM. (Takai duality) Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system with $\mathcal{G}$ a locally compact abelian group. Then

$$\Big(\mathcal{A} \rtimes_\alpha \mathcal{G}\Big) \rtimes_{\hat{\alpha}} \hat{\mathcal{G}} \simeq \mathcal{A} \otimes \mathcal{K}\Big(L^2(\mathcal{G})\Big),$$

where $\mathcal{K}\Big(L^2(\mathcal{G})\Big)$ denotes the compact operators on $L^2(\mathcal{G})$ and $\mathcal{A} \otimes \mathcal{K}\Big(L^2(\mathcal{G})\Big)$ is the subalgebra of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \otimes \mathcal{K}\Big(L^2(\mathcal{G})\Big)$ generated by the appropriate elementary tensors.

New phenomena in non-selfadjoint operator algebras

Recall the definition of the Jacobson Radical of a (not necessarily unital) ring.

DEFINITION. Let $\mathcal{R}$ be a ring. The Jacobson radical $\operatorname{Rad}\mathcal{R}$ is defined as the intersection of all kernels of irreducible representations of $\mathcal{R}$.

In the case where $\mathcal{R}$ is a Banach algebra we have

$$\begin{aligned}\operatorname{Rad}\mathcal{R} &= \{x \in \mathcal{R} \mid \lim_n \|(xy)^n\|^{1/n} = 0, \text{ for all } y \in \mathcal{R}\}\\ &= \{x \in \mathcal{R} \mid \lim_n \|(yx)^n\|^{1/n} = 0, \text{ for all } y \in \mathcal{R}\}.\end{aligned}$$

A ring $\mathcal{R}$ is called semisimple iff $\operatorname{Rad}\mathcal{R} = \{0\}$.

The study of the various radicals is a central topic of investigation in Abstract Algebra and Banach Algebra theory. In Operator Algebras, the Jacobson radical and the semisimplicity of operator algebras have been under investigation since the very beginnings of the theory.

Our next result uncovers a new permanence property in the theory of crossed products.

THEOREM. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system with $\mathcal{G}$ a discrete abelian group. If $\mathcal{A}$ is semisimple then $\mathcal{A} \rtimes_\alpha \mathcal{G}$ is semisimple.

Proof. (For $\mathcal{G} = \mathbb{Z}$) Assume

$$0 \neq a \in \text{Rad}\left(\mathcal{A} \rtimes_\alpha \mathbb{Z}\right).$$

Therefore if

$$a \sim \sum_{n \in \mathbb{Z}} a_n U^n$$

there is a $n \in \mathbb{Z}$ such that $a_n \neq 0$. However

$$a_n U^n = \int_{\mathbb{T}} \hat{\alpha}_t(a) e^{int} dt \in \text{Rad}\left(\mathcal{A} \rtimes_\alpha \mathbb{Z}\right)$$

This implies that $a_n b$ is quasinilpotent for all $b \in \mathcal{A}$ and so $a_n \in \text{Rad}\,\mathcal{A}$.

The previous result raises two natural questions.

(i) Is the converse of the previous Theorem true?

(ii) Is the previous Theorem valid beyond discrete abelian groups?

In order to answer the first question, we investigate a class of operator algebras which was quite popular in the 90's, the (strongly maximal) triangular AF algebras.

DEFINITION. Let $\mathcal{A}$ be a strongly maximal TAF algebra. The dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is said to be *linking* if for every matrix unit $e \in \mathcal{A}$ there exists a group element $g \in G$ such that $e\mathcal{A}\alpha_g(e) \neq \{0\}$.

By Donsig's criterion if $\mathcal{A}$ is semisimple then $(\mathcal{A}, \mathcal{G}, \alpha)$ is linking. The following example shows that there are other linking dynamical systems.

EXAMPLE. Let $\mathcal{A}_n = \mathbb{C} \oplus \mathcal{T}_{2n}$ and define the embeddings $\rho_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by

$$\rho_n(x \oplus a) \;=\; x \oplus \begin{bmatrix} x & & \\ & a & \\ & & x \end{bmatrix}.$$

Then $\mathcal{A} = \varinjlim \mathcal{A}_n$ is a strongly maximal TAF algebra that is not semisimple. Consider the following map $\psi : \mathcal{A}_n \to \mathcal{A}_{n+1}$ given by

$$\psi(x \oplus a) \;=\; x \oplus \begin{bmatrix} x & & \\ & x & \\ & & a \end{bmatrix}.$$

You can see that $\psi \circ \rho_n = \rho_{n+1} \circ \psi$ on $\mathcal{A}_n$ and so ψ is a well-defined map on $\cup \mathcal{A}_n$. By considering that

$$\psi^{-1}(x \oplus a) \;=\; x \oplus \begin{bmatrix} a & & \\ & x & \\ & & x \end{bmatrix}$$

one gets $\psi \circ \psi^{-1} = \psi^{-1} \circ \psi = \rho_{n+1} \circ \rho_n$ on $\mathcal{A}_n$. Hence, ψ extends to be an isometric automorphism of $\mathcal{A}$.

It is easy to see that $(\mathcal{A}, \mathbb{Z}, \psi)$ is a linking dynamical system.

The following theorem and the previous example establish that Question (i) has a negative answer.

THEOREM. Let $\mathcal{A}$ be a strongly maximal TAF algebra and $\mathcal{G}$ a discrete abelian group. The dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ is linking if and only if $\mathcal{A} \rtimes_\alpha G$ is semisimple.

In order to answer the other question we need the following.

LEMMA. Let $\mathcal{A}$ be an operator algebra and let $\mathcal{K}(\mathcal{H})$ denote the compact operators acting on a separable Hilbert space $\mathcal{H}$. If $\mathcal{A} \otimes \mathcal{K}(\mathcal{H})$ is semisimple, then $\mathcal{A}$ is semisimple.

We now show that the semisiplicity theorem does not necessarily hold for groups which are not discrete and abelian. Using our Takai duality, we can actually show that this fails even for $\mathbb{T}$.

EXAMPLE. *A dynamical system* $(\mathcal{B}, \mathbb{T}, \beta)$, *with* $\mathcal{B}$ *a semisimple operator algebra, for which* $\mathcal{B} \rtimes_\beta \mathbb{T}$ *is not semisimple.*

We will employ again our previous results and Takai duality. In the Example we saw a linking dynamical system $(\mathcal{A}, \mathbb{Z}, \alpha)$ for which $\mathcal{A}$ is not semisimple. Since $(\mathcal{A}, \mathbb{Z}, \alpha)$ is linking, we have that the algebra $\mathcal{B} \equiv \mathcal{A} \rtimes_\alpha \mathbb{Z}$ is semisimple. Let $\beta \equiv \widehat{\alpha}$. Then,

$$\mathcal{B} \rtimes_\beta \mathbb{T} = \left(\mathcal{A} \rtimes_\alpha \mathbb{Z}\right) \rtimes_{\widehat{\alpha}} \mathbb{T} \simeq \mathcal{A} \rtimes \mathcal{K}(\ell^2(\mathbb{Z})),$$

which is not semismple,

Nevertheless we have a positive result

THEOREM. Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system, with $\mathcal{G}$ a compact, second countable abelian group. If $\mathcal{A} \rtimes_{\alpha} \mathcal{G}$ is semisimple, then $\mathcal{A}$ is semisimple.

Proof. Assume that $\mathcal{A} \rtimes_{\alpha} \mathcal{G}$ is semisimple. Then the semisimplicity theorem implies that $\Big(\mathcal{A} \rtimes_{\alpha} \mathcal{G}\Big) \rtimes_{\widehat{\alpha}} \widehat{\mathcal{G}}$ is semisimple. By Takai duality, $\mathcal{A} \otimes \mathcal{K}\Big(L^2(\mathcal{G})\Big)$ is semisimple and so $\mathcal{A}$ is semisimple, as desired

We are left to wonder:

(i) What about the semisimplicity of crossed products with $\mathbb{R}$?

(ii) What really goes wrong with the implication

$\mathcal{A} \rtimes_\alpha \mathcal{G}$ is semisimple, then $\mathcal{A}$ is semisimple

in the case where $\mathcal{G}$ is discrete?

Let $\mathcal{G}$ be a *second countable* locally compact abelian group. Let $\Sigma \subseteq \mathcal{G}$ be a *cone* in $\mathcal{G}$ satisfying

(i) $\Sigma \cap \Sigma^{-1} = \{1\}$

(ii) $\Sigma \cdot \Sigma \subseteq \Sigma$

(iii) Σ equals the closure of its interior.

We say that the pair $(\mathcal{G}, \Sigma)$ forms an *ordered abelian group.*. The compact operators leaving invariant all subspaces of $L^2(\mathcal{G}, \mu)$ of the form $L^2(E)$, $E \subseteq \mathcal{G}$ increasing set, are denoted as $\mathcal{K}(\mathcal{G}, \Sigma, m)$.

THEOREM. If $(\mathcal{A}, \mathcal{G}, \Sigma, \alpha)$ is an ordered dynamical system, then we have a stable isomorphism

$$\mathcal{A} \rtimes_\alpha \Sigma \sim_s \left(\mathcal{A} \otimes \mathcal{K}(\mathcal{G}, \Sigma, \mu)\right) \rtimes_{\alpha \otimes \mathsf{Ad}\,\rho} \mathcal{G}.$$

COROLLARY. Let X be a compact metrizable space and ϕ a homeomorphism of X. Then the following are equivalent

(i) $\Big(C(X) \otimes \mathcal{K}^{+}(\mathbb{Z})\Big) \rtimes_{\phi \otimes \mathsf{Ad}\, U} \mathbb{Z}$ is semisimple

(ii) $C(X) \rtimes_{\phi} \mathbb{Z}^{+}$ is semisimple

(iii) recurrent points of (X, ϕ) are dense in X.

Note that $C(X) \otimes \mathcal{K}^{+}(\mathbb{Z})$ is *never* semisimple

An application to C*-algebra theory

Let $(X, \mathcal{C}, \phi)$ be a C*-correspondence and consider

- $\mathcal{T}_X$ the Cuntz-Pimsner-Toeplitz algebra

- $\mathcal{O}_X$ the associated Cuntz Pimsner algebra

- $\mathcal{T}_X^+$ the tensor algebra (Muhly and Solel)

A useful result

THEOREM (Katsoulis and Kribs, 06) If $(X, \mathcal{C}, \phi)$ is a C*-correspondence, then

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_X^+) \simeq \mathcal{O}_X$$

Let $\alpha : \mathcal{G} \to \mathrm{Aut}\,\mathcal{O}_X$ be a generalized gauge action , i.e., a continuous group action with

$$\alpha(\mathcal{C}) = \mathcal{C} \text{ and } \alpha(X) = X$$

THEOREM. (Hao and Ng) Let $(X, \mathcal{C})$ be a non-degenerate C*-correspondence and let $\alpha : \mathcal{G} \to (X, \mathcal{C})$ be a generalized gauge action of a locally compact amenable group $\mathcal{G}$. Then $\mathcal{O}_X \rtimes_{\alpha}^{r} \mathcal{G} \simeq \mathcal{O}_{X \rtimes_{\alpha}^{r} \mathcal{G}}$ via a $*$-isomorphism that maps generators to generators.

QUESTION. What about non-amenable groups?

THEOREM. (E. Bedos, S. Kaliszewski, J. Quigg and D. Robertson, 2015) Let $(X, \mathcal{C})$ be a non-degenerate C*-correspondence and let $\alpha : \mathcal{G} \to (X, \mathcal{C})$ be a generalized gauge action of a discrete and exact group $\mathcal{G}$. Then $\mathcal{O}_X \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{O}_{X \rtimes_\alpha^r \mathcal{G}}$ via a $*$-isomorphism that maps generators to generators.

THEOREM. (Katsoulis, 2016) Let $(X, \mathcal{C})$ be a non-degenerate C*-correspondence and let $\alpha : \mathcal{G} \to (X, \mathcal{C})$ be a generalized gauge action of any discrete group $\mathcal{G}$. Then $\mathcal{O}_X \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{O}_{X \rtimes_\alpha^r \mathcal{G}}$ via a $*$-isomorphism that maps generators to generators.

THEOREM. (Katsoulis, 2016) Let $(X, \mathcal{C})$ be a non-degenerate huperrigid C*-correspondence and let $\alpha : \mathcal{G} \to (X, \mathcal{C})$ be a generalized gauge action of a locally compact exact group $\mathcal{G}$. Then $\mathcal{O}_X \rtimes_\alpha \mathcal{G} \simeq \mathcal{O}_{X \widehat{\rtimes}_\alpha \mathcal{G}}$ via a $*$-isomorphism that maps generators to generators.

A Challenge

An operator algebra is said to be semi-Dirichlet if

$$\mathcal{A}^*\mathcal{A} \subseteq \overline{\mathcal{A} + \mathcal{A}^*}$$

QUESTION. Is there a semi-Dirichlet algebra which is not a tensor algebra of a C*-correspondence? (Davidson and Katsoulis)

Kakariadis constructed a Dirichlet algebra which is not a tensor algebra

REVISED QUESTION. Is there a semi-Dirichlet algebra which is neither a Dirichlet algebra nor the tensor algebra of a C*-correspondence? (Davidson)

EXAMPLE. (A Dirichlet algebra which is not a tensor algebra.)

Consider

$$\mathbb{A}(\mathbb{D}) \rtimes_\alpha \mathbb{Z}$$

where α has two fixed points on $\mathbb{T}$.

This adds a totally new class of examples to Kakariadis' previous counterexamples.

EXAMPLE. (A semi-Dirichlet algebra which is neither a tensor algebra nor a Dirichlet algebra.)

Biholomorphisms of $\overline{\mathbb{B}}_n$ give automorpisms of Popescu's non commutative disc algebra $\mathcal{O}_n^+$.

Consider

$$\mathcal{O}_n^+ \rtimes_\alpha \mathbb{Z}$$

where α is a biholomorphism of $\overline{\mathbb{B}}_n$ with exactly two fixed points on $\theta\mathbb{B}_n$.