Semigroup actions on operator algebras

Evgenios Kakariadis

Newcastle University

Multivariable Operator Theory at the Technion, June 2017

Davidson KR, Fuller AH, Kakariadis ETA, *Semicrossed products of operator algebras by semigroups.*, Memoirs Amer. Math. Soc. **247**, No. 1168 (2017), (97 + v pages).

Kakariadis ETA, *On Nica-Pimsner algebras of C*-dynamical systems over $\mathbb{Z}_+^n$.* , to apear in International Mathematics Research Notices.

I. Framework

In this talk we focus on encoding:

$$\{ \text{ C*-dynamical systems } \} \quad \leftrightsquigarrow \quad \{ \text{ Operator algebras } \}$$

- Origins: Murray, von Neumann (1936, 1940) – Type I, II, and III factors.

- *C*-crossed products*: are constructed based on a given group action $\alpha \colon G \to \mathrm{Aut}(A)$ on a C*-algebra A by *-automorphisms.

- We turn our focus to semigroup actions $\alpha \colon P \to \mathrm{End}(A)$ on a C*-algebra A by **-endomorphisms*.

- Case example: $P = \mathbb{Z}_+$.

I. Framework

Definition

A *C^*-dynamical system* $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$ consists of a *-endomorphism $\alpha\colon A \to A$ of a C*-algebra A.

- Use operators to encode the evolution of the system (in discrete time):

a	$\alpha(a)$	$\alpha^2(a)$	$\cdots$
$t=0$	$t=1$	$t=2$	

- The key is to introduce an "external" operator V that satisfies the *covariance relation*

$$a \cdot V = V \cdot \alpha(a) \text{ for all } a \in A.$$

- Sometimes we would like to impose that V is an isometry (so that positive points in time are reversed).

I. Framework

Example of a unitary pair

For $A \subseteq \mathscr{B}(H)$ and $\alpha \in \mathrm{Aut}(A)$ we have a pair (π, U) on $\ell^2(H)$:

U: bilateral shift $\quad \pi(a) = \mathrm{diag}\{\alpha^n(a) : n \in \mathbb{Z}\} \quad$ s.t. $\quad \pi(a)U = U\pi\alpha(a)$.

Notice that U also "undoes" α: $\pi(a) = U\pi\alpha(a)U^*$.

Gauge Invariant Uniqueness Theorem

Every unitary pair (π', U') that admits a gauge action with π' injective lifts to a faithful representation of $\mathrm{C}^*(\pi(A), U)$.

- In other words $\mathrm{C}^*(\pi(A), U) \simeq A \rtimes \mathbb{Z}$.

Question

What is the analogue for $\alpha \in \mathrm{End}(A)$?

II. Operator algebras over $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Cuntz-Pimsner $\mathscr{O}_{(A,\alpha)}$ (Katsura)

Universal C*-algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^* V = I$), and

$$a \cdot (I - VV^*) = 0, \qquad \text{for} \qquad a \in \ker \alpha^{\perp} := \{a \in A \mid a \cdot \ker \alpha = (0)\}.$$

Remarks

1. If $\alpha \in \mathrm{Aut}(A)$ then $\ker \alpha^{\perp} = A$.Thus V is unitary and $\mathscr{O}_{(A,\alpha)} \simeq A \rtimes_{\alpha} \mathbb{Z}$.
2. When α is injective then $\mathscr{O}(A,\alpha) \simeq A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ for

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & A & \longrightarrow & A_\infty \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_\infty} \\ A & \xrightarrow{\alpha} & A & \longrightarrow & A_\infty \end{array}$$

(Minimal automorphic extension)

II. Operator algebras over $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Cuntz-Pimsner $\mathscr{O}_{(A,\alpha)}$ (Katsura)

Universal C*-algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \qquad \text{for} \qquad a \in \ker\alpha^\perp := \{a \in A \mid a \cdot \ker\alpha = (0)\}.$$

Remarks (Katsura)

3. $\ker\alpha^\perp$ is the largest ideal where the restriction of α is injective.
4. Notice that $a = V\alpha(a)V^*$ for all $a \in \ker\alpha^\perp$.
5. $A \hookrightarrow \mathscr{O}(A,\alpha)$ (Katsura 2004). In fact $\mathscr{O}(A,\alpha)$ is the smallest for which $aV = V\alpha(a)$, V is an isometry and $A \hookrightarrow \mathscr{O}(A,\alpha)$.
6. $\mathscr{O}(A,\alpha)$ satisfies the GIUT.

II. Operator algebras over $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Cuntz-Pimsner $\mathscr{O}_{(A,\alpha)}$ (Katsura)

Universal C*-algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \qquad \text{for} \qquad a \in \ker\alpha^\perp := \{a \in A \mid a \cdot \ker\alpha = (0)\}.$$

Remarks (Katsura)

7. A is exact if and only if $\mathscr{O}(A,\alpha)$ is exact.
8. If A is nuclear then $\mathscr{O}(A,\alpha)$ is nuclear (but not the converse).
9. If $\mathscr{O}(A,\alpha)$ is nuclear then $A/\ker\alpha^\perp$ is nuclear and $\ker\alpha^\perp \hookrightarrow \mathscr{O}(A,\alpha)^\beta$ is nuclear.

II. Operator algebras over $\alpha \colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Cuntz-Pimsner $\mathcal{O}_{(A,\alpha)}$ (Katsura)

Universal C*-algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \qquad \text{for} \qquad a \in \ker\alpha^\perp := \{a \in A \mid a \cdot \ker\alpha = (0)\}.$$

Remarks (Katsoulis-Kribs)

10. $\mathcal{O}(A,\alpha)$ is the C*-envelope in the sense of Arveson for a natural nonselfadjoint operator algebra. Let's pause for a second and put this result into some context.

Intermission: on the C*-envelope

Operator algebras are subalgebras of $\mathscr{B}(H)$

1. Selfadjoint norm-closed subalgebras, i.e. *C*-algebras*.
2. Non-involutive, i.e. *nonselfadjoint operator algebras (nsa)*.

By definition every nsa $\mathscr{A} \subseteq \mathscr{B}(H)$ generates a C*-algebra $\mathrm{C}^*(\mathscr{A})$

It may happen that $\iota_1 \colon \mathscr{A} \to \mathscr{B}(H_1)$ and $\iota_2 \colon \mathscr{A} \to \mathscr{B}(H_2)$ but

$$\mathrm{C}^*(\iota_1(\mathscr{A})) \not\simeq \mathrm{C}^*(\iota_2(\mathscr{A})).$$

Example

The disc algebra $\mathbb{A}(\mathbb{D})$ generates the Toeplitz algebra, $C(\overline{\mathbb{D}})$, and $C(\mathbb{T})$. However $C(\mathbb{T})$ is the *minimal C*-algebra generated by* $\mathbb{A}(\mathbb{D})$, and we call $C(\mathbb{T})$ *the C*-envelope of* $\mathbb{A}(\mathbb{D})$.

Intermission: on the C*-envelope

Question, Arveson (1969)

Does every nsa have a C*-envelope?

Answer: Yes

$\exists\ \iota\colon \mathscr{A} \to \mathscr{B}(H)$ s.t. for any other $\iota'\colon \mathscr{A} \to \mathscr{B}(K)$, $\exists$ a $*$-epimorphism $\Phi\colon \mathrm{C}^*(\iota'(\mathscr{A})) \to \mathrm{C}^*(\iota(\mathscr{A}))$ with $\Phi\iota'(a) = \iota(a)$, $\forall a \in \mathscr{A}$.

The $\mathrm{C}^*(\iota(\mathscr{A}))$ is *the C*-envelope of* $\mathscr{A}$. We write $\mathrm{C}^*_{\text{env}}(\mathscr{A}) = \mathrm{C}^*(\iota(\mathscr{A}))$.

Proofs by:

1. Hamana (1979): $\mathrm{C}^*_{\text{env}}(\mathscr{A})$ is generated in *the injective envelope*.
2. Dritschel-McCullough (2001): $\mathrm{C}^*_{\text{env}}(\mathscr{A})$ is generated by a *maximal dilation*.

Arveson's Program on the C*-envelope

Determine[1] and examine[2] the C*-envelope of a given nsa.

Intermission: on the C*-envelope

Dilations

Let $T \in \mathscr{B}(H)$. *A power dilation* $U \in \mathscr{B}(K)$ *of* T is of the form

$$U = \begin{bmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{bmatrix}.$$

A dilation is *maximal* if it has only trivial dilations.

Example

If T is a contraction ($\|T\| \leq 1$), then the maximal dilation is achieved by a unitary U ($U^*U = UU^* = I$).

Dilations

The idea is that by dilating we obtain "better-behaved" objects.

II. Operator algebras over $\alpha \colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Going back to $\alpha \in \mathrm{End}(A)$.

Semicrossed product $A \times_\alpha \mathbb{Z}_+$

Universal **nonselafdjoint** operator algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$ and V is a **contraction**.

Remark

Inititated by Arveson (1967), formally defined by Peters (1984).

Theorem (Muhly-Solel 2006)

The scp $A \times_\alpha \mathbb{Z}_+$ coincides with the **nsa** *generated by A and V such that $a \cdot V = V \cdot \alpha(a)$ and V is an* **isometry**.

Theorem (Katsoulis-Kribs 2005)

The C-envelope of $A \times_\alpha \mathbb{Z}_+$ is $\mathscr{O}_{(A,\alpha)}$.*

II. Operator algebras over $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Cuntz-Pimsner $\mathscr{O}_{(A,\alpha)}$ (Katsura)

Universal C*-algebra generated by A and V such that $a \cdot V = V \cdot \alpha(a)$, V is an isometry ($V^*V = I$), and

$$a \cdot (I - VV^*) = 0, \qquad \text{for} \qquad a \in \ker\alpha^{\perp} := \{a \in A \mid a \cdot \ker\alpha = (0)\}.$$

Question

Why such complexity?

Remark

1. Let a faithful $\rho\colon A \to \mathscr{B}(H)$ and an isometry V s.t. $\rho(a)V = V\rho\alpha(a)$.
2. If $\rho(a_0) + \sum_{n>0} V_n \rho(a_s) V_n^* = 0$ then $a_0 \in \ker\alpha^{\perp}$ (Katsura 2004).
3. This happens because such equations transform in $\rho(a_0)(I - VV^*) = 0$.

II. Operator algebras over $\alpha \colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Question

Can we connect $\mathscr{O}(A,\alpha)$ to a C*-crossed product in any case?

Two-step dilation (K. 2011)

1. Add a tail: From (A,α) construct an injective (B,β) given by

$$\overset{\alpha}{\circlearrowright} A \xrightarrow{q} A/\ker\alpha^{\perp} \xrightarrow{\mathrm{id}} A/\ker\alpha^{\perp} \xrightarrow{\mathrm{id}} \cdots$$

2. Extend by a direct limit: For (B,β) form the minimal automorphic extension (B_∞,β_∞).

II. Operator algebras over $\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A)$

Theorem (K. 2011)

$$\begin{array}{ccc}
\alpha\colon \mathbb{Z}_+ \to \mathrm{End}(A) & \longrightarrow & \mathscr{O}_{(A,\alpha)} \\
\Big\downarrow \text{dilation} & & \wr\ \text{strong Morita equivalent} \\
\beta_\infty\colon \mathbb{Z} \to \mathrm{Aut}(B_\infty) & \longrightarrow & \mathscr{O}_{(B_\infty,\beta_\infty)} \simeq B_\infty \rtimes_{\beta_\infty} \mathbb{Z}
\end{array}$$

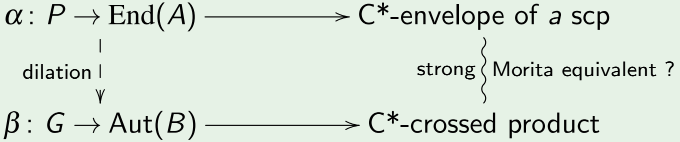
Corollary (K. 2011)

Let $A = C(X)$. TFAE:

1. (A,α) *is minimal and* $\alpha^n \neq \alpha^m$ *for all* $n, m \in \mathbb{Z}_+$*;*
2. (B_∞, β_∞) *is minimal and* $\beta_\infty^n \neq \mathrm{id}$ *for all* $n \in \mathbb{Z}$ *(topol. free);*
3. $B_\infty \rtimes_{\beta_\infty} \mathbb{Z}$ *is simple;*
4. $\mathscr{O}_{(A,\alpha)}$ *is simple (has no non-trivial two-sided closed ideals).*

III. Program on semigroup actions

Question 1



Question 2

Is the C*-envelope a Cuntz-type C*-algebra? Can we describe it by $*$-algebraic relations?

Applications 3

Relate the intrinsic properties of $\alpha\colon P \to \mathrm{End}(A)$ to C*-properties of the obtained object.

III. Program on semigroup actions

Davidson-Fuller-K. (2014)

$$
\begin{array}{ccc}
\alpha\colon P \to \mathrm{End}(A) & \longrightarrow & \text{C*-envelope of a sem. prod.} \\
\text{dilation} \downarrow & & \text{strong} \big\} \text{ Morita equivalent} \\
\beta\colon G \to \mathrm{Aut}(B) & \longrightarrow & \text{C*-crossed product}
\end{array}
$$

1. We confirm this when P is $\mathbb{Z}_+^n$, $\mathbb{F}_n^+$, a spanning cone, an Ore sgrp.
2. For $P = \mathbb{Z}_+^n$ we identify the Cuntz-Nica-Pimsner algebra.
3. We study the Cuntz-Nica-Pimsner algebras in terms of ideal structure.

K. (2014)

4. We study the Nica-Pimsner algebras in terms of nuclearity, exactness, KMS states.

III. Operator algebras over $\alpha \colon \mathbb{Z}_+^n \to \mathrm{End}(A)$

Notation

We write $\mathbf{i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ for all $i = 1, \ldots, n$.

Thus $\alpha \colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ is defined by n commuting $\alpha_{\mathbf{i}} \in End(A)$.

Requirements

1. n contractions $V_{\mathbf{i}}$ such that $a \cdot V_{\mathbf{i}} = V_{\mathbf{i}} \cdot \alpha_{\mathbf{i}}(a)$.
2. The $V_{\mathbf{i}}$ commute.

Is this enough?

The aim is to reach a crossed product. For $A = \mathbb{C}$ we would like to dilate the $V_{\mathbf{i}}$ to unitaries. Parrott's counterexample shows that this cannot be done for any n.

3. We focus on doubly commuting $V_{\mathbf{i}}$, i.e. $V_{\mathbf{i}} V_{\mathbf{j}}^* = V_{\mathbf{j}}^* V_{\mathbf{i}}$ for $i \neq j$.

III. Operator algebras over $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$

The Nica-covariant semicrossed product $A \times_\alpha^{nc} \mathbb{Z}_+^n$ (no involution)

Universal **nonselafdjoint** generated by

$$V_s a, \text{ with } a \in A,\ s \in \mathbb{Z}_+^n,$$

for n doubly commuting **contractions** $V_{\mathbf{i}}$ with $a \cdot V_{\mathbf{i}} = V_{\mathbf{i}} \cdot \alpha_{\mathbf{i}}(a)$.

Remark

A embeds in $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$.

Example

For $A \subseteq H$ let $K = H \otimes \ell^2(\mathbb{Z}_+^n)$ and define

$$S_{\mathbf{i}}(\xi \otimes e_s) = \xi \otimes e_{\mathbf{i}+s} \text{ and } \pi(a)(\xi \otimes e_s) = \alpha_s(a)\xi \otimes e_s$$

for all $s \in \mathbb{Z}_+^n$ and $\xi \in H$. Then π is a faithful representation of A.

III. Operator algebras over $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$

Question

Why do we call it *Nica* covariant?

Theorem (Davidson-Fuller-K. 2014)

The Nc-scp $A \times_\alpha^{nc} \mathbb{Z}_+^n$ *coincides with the* **nsa** *generated by doubly commuting* **isometries** $V_{\mathbf{i}}$ *and* A *such that* $a \cdot V_{\mathbf{i}} = V_{\mathbf{i}} \cdot \alpha_{\mathbf{i}}(a)$.

Remark

Doubly commuting isometries form a representation of $\mathbb{Z}_+^n$ in the sense of Nica.

Corollary

If $\mathrm{C}^*_{\text{env}}(A \times_\alpha^{nc} \mathbb{Z}_+^n)$ *is generated by* A *and* V *then*

$$\mathrm{C}^*_{\text{env}}(A \times_\alpha^{\text{nc}} \mathbb{Z}_+^n) \simeq \overline{\mathrm{span}}\{V_s a V_t^* : a \in \mathscr{A} \text{ and } s, t \in \mathbb{Z}_+^n\}.$$

III. Reductions

The plan

Dilate a system $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ to a group action $\beta\colon \mathbb{Z}^n \to \mathrm{Aut}(B)$.

Injective case: $\ker \alpha_{\mathbf{i}} = (0)$ for all $i = 1, \ldots, n$.

We can then construct the direct limit $\beta_{\mathbf{i}} \in \mathrm{Aut}(B)$ s.t.

$$
\begin{array}{ccccc}
A_s & \xrightarrow{\alpha_t} & A_{s+t} & \longrightarrow & B \\
\downarrow{\scriptstyle \alpha_{\mathbf{i}}} & & \downarrow{\scriptstyle \alpha_{\mathbf{i}}} & & \downarrow{\scriptstyle \beta_{\mathbf{i}}} \\
A_s & \xrightarrow{\alpha_t} & A_{s+t} & \longrightarrow & B
\end{array}
$$

where $A_s = A$ for all $s \in \mathbb{Z}_+^2$.

Then $\mathrm{C}^*_{\mathrm{env}}(A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n) \simeq B \rtimes_\beta \mathbb{Z}^n$ (Corollary Davidson-Fuller-K. 2014).

III. Reductions

The (revised) plan

Dilate a system $\alpha : \mathbb{Z}_+^n \to \mathrm{End}(A)$ where $\ker \alpha_{\mathbf{i}} \neq (0)$ to a system $\beta : \mathbb{Z}_+^n \to \mathrm{End}(B)$ such that $\ker \beta_{\mathbf{i}} = (0)$.

The $n = 1$ case (K. 2011)

For $I = \ker \alpha^{\perp}$ let $B = A \oplus c_0(A/I)$ and $\beta(a, (x_n)) = (\alpha(a), a + I, (x_n))$.

$$\overset{\alpha}{\circlearrowright} A \xrightarrow{q_I} A/I \xrightarrow{\mathrm{id}} A/I \xrightarrow{\mathrm{id}} \cdots$$

The $n = 2$ case

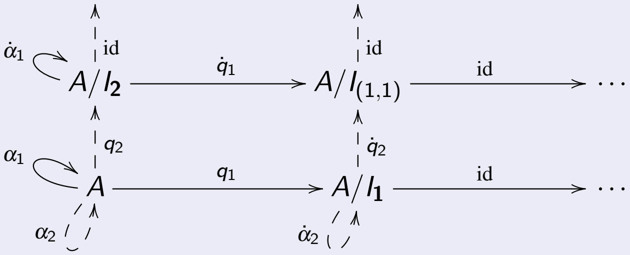
Let $\alpha_{\mathbf{1}}, \alpha_{\mathbf{2}} \in \mathrm{End}(A)$ such that $\alpha_{\mathbf{1}}\alpha_{\mathbf{2}} = \alpha_{\mathbf{2}}\alpha_{\mathbf{1}}$. We want two injective commuting $\beta_{\mathbf{1}}, \beta_{\mathbf{2}}$ on some $B \supseteq A$ that dilate $\alpha_{\mathbf{1}}, \alpha_{\mathbf{2}}$.

III. Non-injective case

A first attempt

Let $I_{(1,1)} := (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp}$, $I_{\mathbf{1}} := \cap_n \alpha_{\mathbf{2}}^{-n}(I_{(1,1)})$, $I_{\mathbf{2}} := \cap_n \alpha_{\mathbf{1}}^{-n}(I_{(1,1)})$.

Let $\beta_{\mathbf{1}}$ be the solid arrows and $\beta_{\mathbf{2}}$ the broken arrows:



with $\dot{\alpha}_{\mathbf{1}} q_{\mathbf{2}} = q_{\mathbf{1}} \alpha_{\mathbf{2}}$ and $\dot{q}_{\mathbf{1}} q_{\mathbf{1}} = q_{(1,1)}$ (plus the symmetrical ones).

Then β is injective and generalises the $n = 1$ case.

However this construction is bound to fail!

III. Non-injective case

How did we end up with $I_{(1,1)} = (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp}$?

1. Let a faithful $\rho \colon A \to \mathscr{B}(H)$ and doubly commuting isometries $V_{\mathbf{i}}$ such that
$$\rho(a) V_{\mathbf{i}} = V_{\mathbf{i}} \rho \alpha_{\mathbf{i}}(a).$$
2. Because of a gauge action, we will have to deal with equations
$$\rho(a_0) + \sum_{s > \underline{0}} V_s \rho(a_s) V_s^* = 0.$$
3. This transforms into
$$\rho(a_0)(I - V_{\mathbf{1}} V_{\mathbf{1}}^*)(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0.$$
4. From this we get that $a_0 \perp \ker \alpha_{\mathbf{1}}, \ker \alpha_{\mathbf{2}}$.

III. Non-injective case

Why isn't $I_{(1,1)} = (\ker\alpha_{\mathbf{1}} \cdot \ker\alpha_{\mathbf{2}})^{\perp}$ enough?

However we will also have equations of the form

$$\rho(a_0) + \sum_{n > \underline{0}} V_{(n,0)}\rho(a_n)V_{(n,0)}^{*} = 0$$

which transform into

$$\rho(a_0)(I - V_{\mathbf{1}}V_{\mathbf{1}}^{*}) = 0.$$

From this we get that $a_0 \perp \ker\alpha_{\mathbf{1}}$.

From this we also get that $\alpha_{(0,n)}(a_0) \perp \ker\alpha_{\mathbf{1}}$ for all $n > 0$.

This happens because $\rho\alpha_{\mathbf{2}}(a) = V_{\mathbf{2}}^{*}\rho(a)V_{\mathbf{2}}$.

So we need the ideal $I_{\mathbf{1}} = \cap_n \alpha_{\mathbf{2}}^{-n}(\ker\alpha_{\mathbf{1}}^{\perp})$ instead of $\cap_n \alpha_{\mathbf{2}}^{-n}(I)$.

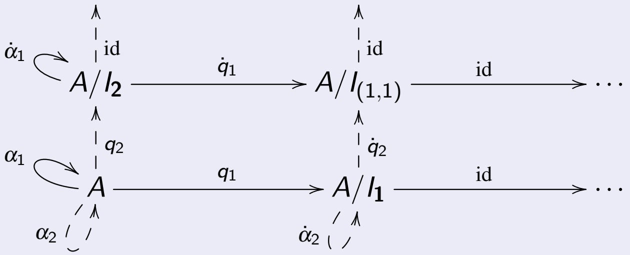
And of course its symmetrical $I_{\mathbf{2}}$.

III. Non-injective case

Correct tail

$I_{(1,1)} = (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp} \quad I_{\mathbf{1}} = \cap_n \alpha_{\mathbf{2}}^{-n}(\ker \alpha_{\mathbf{1}}^{\perp}) \quad I_{\mathbf{2}} = \cap_n \alpha_{\mathbf{1}}^{-n}(\ker \alpha_{\mathbf{2}}^{\perp})$.

Then define $\beta_{\mathbf{1}}$ and $\beta_{\mathbf{2}}$ by



with $\dot{\alpha}_{\mathbf{1}} q_{\mathbf{2}} = q_{\mathbf{1}} \alpha_{\mathbf{2}}$ and $\dot{q}_{\mathbf{2}} q_{\mathbf{1}} = q_{(1,1)}$ (plus the symmetrical ones).

Then $\beta_{\mathbf{1}}$ and $\beta_{\mathbf{2}}$ generalise the $n = 1$ case.

It is not immediate but they are commuting and injective.

III. General construction

For $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_+^n$, define

$$\mathrm{supp}(\underline{x}) = \{\mathbf{i} : x_i \neq 0\} \text{ and } \underline{x}^\perp = \{\underline{y} \in \mathbb{Z}_+^n : \mathrm{supp}(\underline{y}) \cap \mathrm{supp}(\underline{x}) = \emptyset\}$$

and let the ideals

$$I_{\underline{x}} = \bigcap_{\underline{y} \in \underline{x}^\perp} \alpha_{\underline{y}}^{-1}\Big(\big(\bigcap_{\mathbf{i} \in \mathrm{supp}(\underline{x})} \ker \alpha_{\mathbf{i}}\big)^\perp\Big).$$

Let $B_{\underline{x}} = A/I_{\underline{x}}$ and on the C*-algebra

$$B = \sum_{\underline{x} \in \mathbb{Z}_+^n}^{\oplus} B_{\underline{x}}$$

define the $*$-endomorphisms

$$\beta_{\mathbf{i}}(q_{\underline{x}}(a) \otimes e_{\underline{x}}) = \begin{cases} q_{\underline{x}}\alpha_{\mathbf{i}}(a) \otimes e_{\underline{x}} + q_{\underline{x}+\mathbf{i}}(a) \otimes e_{\underline{x}+\mathbf{i}} & \text{for } \mathbf{i} \in \underline{x}^\perp, \\ q_{\underline{x}}(a) \otimes e_{\underline{x}+\mathbf{i}} & \text{for } \mathbf{i} \in \mathrm{supp}(\underline{x}). \end{cases}$$

Then the $\beta_{\mathbf{i}}$ commute and are injective (this is not trivial).

III. C*-envelope

Theorem (Davidson-Fuller-K. 2014)

Let $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ *be a semigroup action. Apply the constructions:*

1. *dilate* (A,α) *to the injective* (B,β) *by adding a tail;*
2. *use the direct limit to extend it to* $\beta_\infty\colon \mathbb{Z}^n \to \mathrm{Aut}(B_\infty)$*.*

Then the C-envelope of* $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$ *is Morita equivalent to* $B_\infty \rtimes_{\beta_\infty} \mathbb{Z}^n$*.*

Remarks

1. The C*-envelope is defined by a co-universal property for which

$$\mathrm{C}^*_{\mathrm{env}}(A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n) \simeq \overline{\mathrm{span}}\{V_s a V_t^* : a \in \mathscr{A} \text{ and } s,t \in \mathbb{Z}_+^n\}.$$

2. This was one of the challenging points in the proof.

What about the structure of the C*-envelope?

Can we identify the C*-envelope by C*-algebraic relations?

III. Towards a Cuntz algebra

Recall

For $n = 2$ we arrived to the equalities

1. $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*) = 0$;
2. $a(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$;
3. $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*)(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$;

subject to a. Then we used the solutions/ideals to produce the tail. This appears to be more than an innocent coincidence.

The Cuntz-Nica-Pimsner algebra for $n = 2$ case

It is the universal C*-algebra such that: (a) $V_{\mathbf{i}}$ are doubly commuting isometries; (b) $aV_{\mathbf{i}} = V_{\mathbf{i}}\alpha_{\mathbf{i}}(a)$; and (c) we have

c.1 $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*) = 0$ for all $a \in \cap_n \alpha_{\mathbf{2}}^{-n}(\ker \alpha_{\mathbf{1}}^{\perp})$;

c.2 $a(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$ for all $a \in \cap_n \alpha_{\mathbf{1}}^{-n}(\ker \alpha_{\mathbf{2}}^{\perp})$;

c.3 $a(I - V_{\mathbf{1}} V_{\mathbf{1}}^*)(I - V_{\mathbf{2}} V_{\mathbf{2}}^*) = 0$ for all $a \in (\ker \alpha_{\mathbf{1}} \cdot \ker \alpha_{\mathbf{2}})^{\perp}$.

III. The Cuntz-Nica-Pimsner algebra

Definition (Davidson-Fuller-K. 2014)

The *Cuntz-Nica-Pimsner algebra* $\mathcal{N}\mathcal{O}(A,\alpha)$ *of* $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ is the universal C*-algebra generated by A and $V_{\mathbf{i}}$ so that:

1. $V_{\mathbf{i}}$ are commuting isometries;
2. $aV_{\mathbf{i}} = V_{\mathbf{i}}\alpha_{\mathbf{i}}(a)$; and
3. $a \cdot \prod_{\mathbf{i}\in \mathrm{supp}(\underline{x})}(I - V_{\mathbf{i}}V_{\mathbf{i}}^*) = 0$ for $a \in \bigcap_{\underline{y}\in\underline{x}^\perp} \alpha_{\underline{y}}^{-1}\Big(\big(\bigcap_{\mathbf{i}\in\mathrm{supp}(\underline{x})} \ker \alpha_{\mathbf{i}}\big)^\perp\Big)$.

Corollary (Davidson-Fuller-K. 2014)

1. *The C*-envelope of* $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$ *is* $\mathcal{N}\mathcal{O}(A,\alpha)$.
2. *For* $\alpha\colon \mathbb{Z}_+^n \to \mathrm{End}(A)$ *there exists a dilation* $\beta_\infty\colon \mathbb{Z}^n \to \mathrm{Aut}(B_\infty)$ *such that* $\mathcal{N}\mathcal{O}(A,\alpha) \overset{\mathrm{sMe}}{\sim} B_\infty \rtimes_{\beta_\infty} \mathbb{Z}^n$.
3. $\mathcal{N}\mathcal{O}(A,\alpha)$ satisfies the GIUT with respect to the CNP representations.

III. Simplicity

Theorem (Davidson-Fuller-K. 2014)

$$\begin{array}{ccc}
\alpha \colon \mathbb{Z}_+^n \to \mathrm{End}(A) & \longrightarrow & \mathcal{N}\mathcal{O}(A,\alpha) \simeq \mathrm{C}^*_{\mathrm{env}}(A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n) \\
\text{dilation} \downarrow & & \text{strong} \wr \text{ Morita equivalent} \\
\beta_\infty \colon \mathbb{Z}^n \to \mathrm{Aut}(B_\infty) & \longrightarrow & B_\infty \rtimes_{\beta_\infty} \mathbb{Z}^n
\end{array}$$

Corollary (Davidson-Fuller-K. 2014)

Let $A = C(X)$ and let $\phi_s \colon X \to X$ related to $\alpha_s \colon X \to X$. TFAE:

1. *(A,α) is minimal and $\{x \in X \mid \phi_s(x) \neq \phi_r(x)\}^o = \emptyset$ for all $s, r \in \mathbb{Z}_+^n$ (top. free);*
2. *(B_∞, β_∞) is minimal and topologically free;*
3. *$B_\infty \rtimes_{\beta_\infty} \mathbb{Z}$ is simple;*
4. *$\mathcal{N}\mathcal{O}_{(A,\alpha)}$ is simple.*

III. Exactness/Nuclearity

Theorem (K. 2014)

$\mathcal{N}\mathcal{O}(A,\alpha)$ is exact if and only if A is exact.

Theorem (K. 2014)

Let $\beta_\infty \colon \mathbb{Z}^n \to \mathrm{Aut}(B_\infty)$ be the automorphic dilation of $\alpha \colon \mathbb{Z}_+^n \to \mathrm{End}(A)$. TFAE:

1. *the embeddings $A, A/I_s \hookrightarrow B_\infty$ are nuclear for all $s \in \mathbb{Z}_+^n$;*
2. *B_∞ is nuclear;*
3. *$B_\infty \rtimes_{\beta_\infty} \mathbb{Z}^n$ is nuclear;*
4. *$\mathcal{N}\mathcal{O}(A,\alpha)$ is nuclear.*

Proposition (K. 2014)

If A is nuclear or if $A \hookrightarrow \mathrm{C}^*(V_{n\mathbf{1}} a V_{n\mathbf{1}}^* \mid a \in A, n \in \mathbb{Z}_+)$ is nuclear then $\mathcal{N}\mathcal{O}(A,\alpha)$ is nuclear. The converse is not true.

IV. Remarks

Remarks on $\mathscr{N}\mathscr{T}(A,\alpha)$ (K. 2014)

1. There is a second variant, the Toeplitz-Nica-Pimsner algebra.

2. For this we get A is nuclear (resp. exact) if and only if $\mathscr{N}\mathscr{T}(A,\alpha)$ is nuclear (resp. exact).

KMS states (K. 2014)

3. The gauge action implements an action of $\mathbb{R}$ on the Nica-Pimsner algebras. We are able to identify all KMS states at finite temperature: for any $T < \infty$ there is exactly one $\mathrm{KMS}_{1/T}$ state.

4. For $T = \infty$ the KMS states are the tracial states and there is no bijection (there might be more than one).

IV. Remarks

Remarks on simplicity

5. Kalantar-Kennedy show that simplicity of the reduced C*-crossed product is equivalent to topological freeness of the group action on a boundary.

6. We are working towards formulating this property for semigroups and showing its stability under the automorphic dilation.

Remarks on product systems

7. Both $\mathcal{N}\mathcal{T}(A,\alpha)$ and $\mathcal{N}\mathcal{O}(A,\alpha)$ are examples of C*-algebras associated to product systems.

8. A gauge invariance uniqueness theorem for general Toeplitz-Nica-Pimsner algebras is easy to obtain by our methods.

9. We believe that the same is true for the Cuntz-Nica-Pimsner algebras (with Adam Dor-On).

The end.