Hyperbolic Geometry on Noncommutative Polyballs

GELU POPESCU

University of Texas at San Antonio

Technion-Israel Institute of Technology, June 2017

Papers

- G. Popescu, Free Pluriharmonic Functions on Noncommutative Polyballs, *Analysis & PDE*, **9** (2016).
- G. Popescu, Hyperbolic Geometry on Noncommutative Polyballs, submitted.

Noncommutative polyballs

- $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ denotes the set of all tuples $\mathbf{X} = (X_1, \ldots, X_k)$ with the property that the entries of $X_s := (X_{s,1}, \ldots, X_{s,n_s})$ are commuting with the entries of $X_t := (X_{t,1}, \ldots, X_{t,n_t})$ for any $s, t \in \{1, \ldots, k\}$, $s \neq t$.

- The open *polyball* :

$$\mathbf{P_n}(\mathcal{H}) := [B(\mathcal{H})^{n_1}]_1 \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_1,$$

where $[B(\mathcal{H})^{n_i}]_1$ is the open unit ball

$$\{(X_{i,1}, \ldots, X_{i,n_i}) \in B(\mathcal{H})^{n_i} : \ \|X_{i,1}X_{i,1}^* + \cdots + X_{i,n_i}X_{i,n_i}^*\| < 1\}.$$

Noncommutative regular polyballs

- The *regular polyball* on the Hilbert space $\mathcal{H}$ is defined by

$$\mathbf{B_n}(\mathcal{H}) := \{\mathbf{X} \in \mathbf{P_n}(\mathcal{H}) : \ \mathbf{\Delta_X}(I) > 0\},$$

where the *defect mapping* $\mathbf{\Delta_X} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\mathbf{\Delta_X} := \left(id - \Phi_{X_1}\right) \circ \cdots \circ \left(id - \Phi_{X_k}\right),$$

and $\Phi_{X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by

$$\Phi_{X_i}(Y) := \sum_{j=1}^{n_i} X_{i,j} Y X_{i,j}^*, \qquad Y \in B(\mathcal{H}).$$

- *(Abstract) regular polyball* $\mathbf{B_n} := \coprod_{\mathcal{H}} \mathbf{B_n}(\mathcal{H})$.

Universal models

- Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e^i_1, \ldots, e^i_{n_i}$. The *full Fock space* of H_{n_i} is defined by

$$F^2(H_{n_i}) := \mathbb{C}1 \oplus \bigoplus_{s\geq 1} H_{n_i}^{\otimes s}.$$

- Let $\mathbb{F}^+_{n_i}$ be the unital free semigroup on n_i generators $g^i_1, \ldots, g^i_{n_i}$ and the identity g^i_0. Set $e^i_\alpha := e^i_{j_1} \otimes \cdots \otimes e^i_{j_p}$ if $\alpha = g^i_{j_1} \cdots g^i_{j_p} \in \mathbb{F}^+_{n_i}$ and $e^i_{g^i_0} := 1 \in \mathbb{C}$.

- For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, the left creation operator $S_{i,j}$ on $F^2(H_{n_i})$ is defined by setting

$$S_{i,j} e^i_\alpha := e^i_j \otimes e^i_\alpha, \qquad \alpha \in \mathbb{F}^+_{n_i}.$$

Universal models

Definition

The operator $\mathbf{S}_{i,j}$ acting on $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ is defined by

$$\mathbf{S}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes S_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}}.$$

- Similarly, we define the *right creation operator* $R_{i,j} : F^2(H_{n_i}) \to F^2(H_{n_i})$ by setting $R_{i,j} e^i_\alpha := e^i_\alpha \otimes e^i_j$ and the corresponding $\mathbf{R}_{i,j}$.
- The noncommutative polyball algebra $\mathcal{A}_\mathbf{n}$ (resp $\mathcal{R}_\mathbf{n}$) is the norm closed non-selfadjoint algebra generated by $\{\mathbf{S}_{i,j}\}$ (resp. $\{\mathbf{R}_{i,j}\}$) and the identity.

Universal models

- The k-tuple $\mathbf{S} := (\mathbf{S}_1, \ldots, \mathbf{S}_k)$, where $\mathbf{S}_i := (\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i})$, is a pure element in the regular polyball $\mathbf{B_n}(\otimes_{i=1}^k F^2(H_{n_i}))^-$ and plays the role of *universal model* for the abstract regular polyball.
- Let $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})$ with $X_i := (X_{i,1}, \ldots, X_{i,n_i})$.
- Set $X_{i,\alpha_i} := X_{i,j_1} \cdots X_{i,j_p}$ if $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $X_{i,g_0^i} := I$.
- If $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, denote $\mathbf{X}_{\boldsymbol{\alpha}} := \mathbf{X}_{1,\alpha_1} \cdots \mathbf{X}_{k,\alpha_k}$.

Main results on free pluriharmonic functions

- Introduce and characterize the class of k-multi-Toeplitz operators on $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$.
- Characterize the bounded free k-pluriharmonic functions and solve the Dirichlet extension problem on regular polyballs.
- Give necessary and sufficient conditions for a function to be the Poisson transform of a completely bounded (resp. completely positive) map on $C^*(\mathbf{S})$, the C^*-algebra generated by the universal model of the polyball.
- Obtain Herglotz-Riesz representation theorems for free holomorphic functions with positive real parts on regular polyballs.

k-multi-Toeplitz operators

- Brown and Halmos (Crelle, 1963) proved :

Theorem

A bounded linear operator T on the Hardy space $H^2(\mathbb{D})$ is a Toeplitz operator if and only if $S^*TS = T$, where S is the unilateral shift.

Definition

A bounded linear operator T on $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ is called *k-multi-Toeplitz operator* with respect to the right universal model $\mathbf{R} = \{\mathbf{R}_{i,j}\}$ if, for each $i \in \{1, \ldots, k\}$,

$$\mathbf{R}_{i,s}^* T \mathbf{R}_{i,t} = \delta_{st} T, \qquad s, t \in \{1, \ldots, n_i\}.$$

k-multi-Toeplitz operators

- Each k-multi-Toeplitz operator T has a uniquely determined formal power series in several variables.
- One can recapture T from its "Fourier series".
- We characterize the noncommutative formal power series which are Fourier series of k-multi-Toeplitz operators.

Theorem

The set of all k-multi-Toeplitz operators on $\bigotimes_{i=1}^{k} F^2(H_{n_i})$ coincides with

$$\mathcal{T}_{\mathbf{n}} := \operatorname{span}\{\mathcal{A}_{\mathbf{n}}^* \mathcal{A}_{\mathbf{n}}\}^{-\mathrm{SOT}} = \operatorname{span}\{\mathcal{A}_{\mathbf{n}}^* \mathcal{A}_{\mathbf{n}}\}^{-\mathrm{WOT}},$$

where $\mathcal{A}_{\mathbf{n}}$ is the noncommutative polyball algebra.

Noncommutative Berezin kernels

- If $\mathbf{X} = \{X_{i,j}\} \in \mathbf{B_n}(\mathcal{H})^-$, define the noncommutative Berezin kernel

$$\mathbf{K_X} : \mathcal{H} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\mathbf{\Delta_X}(I)^{1/2}(\mathcal{H})}$$

by setting

$$\mathbf{K_X} h := \sum_{\beta_i \in \mathbb{F}_{n_i}^+} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes \mathbf{\Delta_X}(I)^{1/2} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^* h,$$

where the defect operator is given by

$$\mathbf{\Delta_X}(I) := (id - \Phi_{X_1}) \circ \cdots \circ (id - \Phi_{X_k})(I).$$

Noncommutative Berezin transforms

- The *Berezin transform* at $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$ is the map $\mathcal{B}_{\mathbf{X}} : B(\otimes_{i=1}^k F^2(H_{n_i})) \to B(\mathcal{H})$ defined by

$$\mathcal{B}_{\mathbf{X}}[g] := \mathbf{K}_{\mathbf{X}}^*(g \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{X}}, \qquad g \in B(\otimes_{i=1}^k F^2(H_{n_i})).$$

- If $g \in C^*(\mathbf{S})$, the C^*-algebra generated by $\mathbf{S}_{i,1}, \ldots, \mathbf{S}_{i,n_i}$, we define the Berezin transform at $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})^-$ by

$$\mathcal{B}_{\mathbf{X}}[g] := \lim_{r \to 1} \mathbf{K}_{r\mathbf{X}}^*(g \otimes I_{\mathcal{H}})\mathbf{K}_{r\mathbf{X}},$$

  where the limit is in the operator norm topology.
- $\mathcal{B}_{\mathbf{X}}$ is a unital completely positive linear map such that

$$\mathcal{B}_{\mathbf{X}}(\mathbf{S}_{\boldsymbol{\alpha}}\mathbf{S}_{\boldsymbol{\beta}}^*) = \mathbf{X}_{\boldsymbol{\alpha}}\mathbf{X}_{\boldsymbol{\beta}}^*, \qquad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+,$$

  where $\mathbf{S}_{\boldsymbol{\alpha}} := \mathbf{S}_{1,\alpha_1} \cdots \mathbf{S}_{k,\alpha_k}$ if $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k)$.

Free k-pluriharmonic functions

Definition

A function F is called free k-pluriharmonic on the polyball $\mathbf{B_n}$ if it has the form

$$F(\mathbf{X}) = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda} a_{\boldsymbol{\alpha},\boldsymbol{\beta}} X_{1,\alpha_1}\cdots X_{k,\alpha_k} X^*_{1,\beta_1}\cdots X^*_{k,\beta_k},$$

where $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda$ iff $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_k)$ and $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_k)$, with $\alpha_i,\beta_i\in\mathbb{F}^+_{n_i}$, $|\alpha_i|=m_i^-$, $|\beta_i|=m_i^+$, and the series converge in the operator norm topology for any $\mathbf{X}=(X_1,\ldots,X_k)\in\mathbf{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$.

- F is bounded if $\|F\| := \sup_{\mathbf{X}\in\mathbf{B_n}(\mathcal{H})}\|F(\mathbf{X})\| < \infty$.

Free k-pluriharmonic functions

- Let $\mathbf{PH}^\infty(\mathbf{B_n})$ be the vector space of all bounded free k-pluriharmonic functions on $\mathbf{B_n}$.
- For each $m = 1, 2, \ldots$, define the norm $\|\cdot\|_m : M_m(\mathbf{PH}^\infty(\mathbf{B_n})) \to [0, \infty)$ by setting

$$\|[F_{ij}]_m\|_m := \sup \|[F_{ij}(\mathbf{X})]_m\|,$$

  where sup is taken over all $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$ and any $\mathcal{H}$.
- The norms $\|\cdot\|_m$ determine an operator space structure on $\mathbf{PH}^\infty(\mathbf{B_n})$, in the sense of Ruan.

Bounded free k-pluriharmonic functions

Theorem

If $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a free k-pluriharmonic function, then the F is bounded if and only if there exists $A \in \mathcal{T}_\mathbf{n}$ such that

$$F(\mathbf{X}) = \mathcal{B}_\mathbf{X}[A] := \mathbf{K}_\mathbf{X}^*(A \otimes I_\mathcal{H})\mathbf{K}_\mathbf{X}, \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

In this case, $A = \text{SOT-}\lim_{r\to 1} F(r\mathbf{S})$.

Moreover, the map

$$\Phi : \mathbf{PH}^\infty(\mathbf{B_n}) \to \mathcal{T}_\mathbf{n} \quad \textit{defined by} \quad \Phi(F) := A$$

is a completely isometric isomorphism of operator spaces.

Dirichlet extension problem for regular polyballs

- Let $\mathbf{PH}^c(\mathbf{B_n})$ be the set of all free k-pluriharmonic functions on $\mathbf{B_n}$ which have continuous extensions to $\mathbf{B_n}(\mathcal{H})^-$ (in norm topology), for any Hilbert space $\mathcal{H}$.
- Assume that $\mathcal{H}$ is an infinite dimensional Hilbert space.

Dirichlet extension problem for regular polyballs

Theorem

If $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a free k-pluriharmonic function, then F has a continuous extension to the closed polyball $\mathbf{B_n}(\mathcal{H})^-$ (in the operator norm) if and only if there exists $A \in \mathcal{P} := \mathrm{span}\{f^*g : \ f, g \in \mathcal{A}_\mathbf{n}\}^{-\|\cdot\|}$ such that

$$F(\mathbf{X}) = \mathcal{B}_\mathbf{X}[A], \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

In this case, $A = \lim_{r\to 1} F(r\mathbf{S})$, where the convergence is in the operator norm. Moreover, the map

$$\Phi : \mathbf{PH}^c(\mathbf{B_n}) \to \mathcal{P} \quad \text{defined by} \quad \Phi(F) := A$$

is a completely isometric isomorphism of operator spaces.

Noncommutative Poisson transforms of c.b. maps

- Consider the operator system

$$\mathcal{R}_{\mathbf{n}}^*\mathcal{R}_{\mathbf{n}} := \operatorname{span}\{\mathbf{R}_{\boldsymbol{\alpha}}^*\mathbf{R}_{\boldsymbol{\beta}} : \ \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\},$$

where $\mathbf{R} := (\mathbf{R}_1, \ldots, \mathbf{R}_k)$ and $\mathbf{R}_i := (\mathbf{R}_{i,1}, \ldots, \mathbf{R}_{i,n_i})$.

- If $\mu : \mathcal{R}_{\mathbf{n}}^*\mathcal{R}_{\mathbf{n}} \to B(\mathcal{E})$ is a completely bounded linear map, then there exists a unique completely bounded linear map

$$\widehat{\mu} := \mu \otimes id : \overline{\mathcal{R}_{\mathbf{n}}^*\mathcal{R}_{\mathbf{n}}}^{\|\cdot\|} \otimes_{min} B(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$$

such that

$$\widehat{\mu}(A \otimes Y) = \mu(A) \otimes Y, \qquad A \in \mathcal{R}_{\mathbf{n}}^*\mathcal{R}_{\mathbf{n}},\ Y \in B(\mathcal{H}).$$

Moreover, $\|\widehat{\mu}\|_{cb} = \|\mu\|_{cb}$ and, if μ is completely positive, then so is $\widehat{\mu}$.

Noncommutative Poisson transforms of c.b. maps

- Define the *free pluriharmonic Poisson kernel* by setting

$$\mathcal{P}(\mathbf{R},\mathbf{X}) := \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda} \mathbf{R}^*_{\tilde{\boldsymbol{\alpha}}}\mathbf{R}_{\tilde{\boldsymbol{\beta}}} \otimes \mathbf{X}_{\boldsymbol{\alpha}}\mathbf{X}^*_{\boldsymbol{\beta}}, \quad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}),$$

  where the convergence is in the operator norm topology, and $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Lambda$ iff $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_k)$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_k)$, with $\alpha_i,\beta_i \in \mathbb{F}^+_{n_i}$, $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$.

- We introduce the *noncommutative Poisson transform of a c. b. map* $\mu : \mathcal{R}^*_{\mathbf{n}}\mathcal{R}_{\mathbf{n}} \to B(\mathcal{E})$ on the regular polyball to be the map $\mathcal{P}\mu : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ defined by

$$(\mathcal{P}\mu)(\mathbf{X}) := \widehat{\mu}[\mathcal{P}(\mathbf{R},\mathbf{X})], \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$

Noncommutative Poisson transforms of c.b. maps

Theorem

Let $\mu : \mathcal{R}_{\mathbf{n}}^* \mathcal{R}_{\mathbf{n}} \to B(\mathcal{E})$ *be a completely bounded linear map. The following statements hold.*

(i) *The map* $\mathbf{X} \mapsto \mathcal{P}(\mathbf{R}, \mathbf{X})$ *is a positive* k*-pluriharmonic function on the polyball* $\mathbf{B_n}$*, with coefficients in* $B(\otimes_{i=1}^k F^2(H_{n_i}))$*, and has the factorization* $\mathcal{P}(\mathbf{R}, \mathbf{X}) = C_{\mathbf{X}}^* C_{\mathbf{X}}$*, where*

$$C_{\mathbf{X}} := (I \otimes \mathbf{\Delta_X}(I)^{1/2}) \prod_{i=1}^{k} \left(I - \mathbf{R}_{i,1} \otimes X_{i,1}^* - \cdots - \mathbf{R}_{i,n_i} \otimes X_{i,n_i}^* \right)^{-1}.$$

(ii) *The noncommutative Poisson transform* $\mathcal{P}\mu$ *is a free* k*-pluriharmonic function on the regular polyball* $\mathbf{B_n}$*.*

Noncommutative Poisson transforms of c.b. maps

(iii) If μ is a completely positive linear map, then $\mathcal{P}\mu$ is a positive free k-pluriharmonic function on $\mathbf{B_n}$.

- Let F be a free k-pluriharmonic function on the polyball $\mathbf{B_n}$, with operator-valued coefficients in $B(\mathcal{E})$, and with representation

$$F(\mathbf{X}) = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda} A_{\boldsymbol{\alpha},\boldsymbol{\beta}} \otimes \mathbf{X}_{\boldsymbol{\alpha}}\mathbf{X}_{\boldsymbol{\beta}}^*.$$

- We associate with F and each $r\in[0,1)$ the linear map $\nu_{F_r} : \mathcal{R}_{\mathbf{n}}^*\mathcal{R}_{\mathbf{n}} \to B(\mathcal{E})$ by setting

$$\nu_{F_r}(\mathbf{R}_{\widetilde{\boldsymbol{\alpha}}}^*\mathbf{R}_{\widetilde{\boldsymbol{\beta}}}) := r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} A_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \qquad (\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda.$$

Noncommutative Poisson transforms of c.b. maps

Theorem

Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ be a free k-pluriharmonic function. Then the following statements are equivalent :

(i) *there exists a completely bounded linear map $\mu : C^*(\mathbf{R}) \to B(\mathcal{E})$ such that $F = \mathcal{P}\mu$;*

(ii) *the linear maps $\{\nu_{F_r}\}_{r\in[0,1)}$ associate with F are completely bounded and $\sup\limits_{0\leq r<1} \|\nu_{F_r}\|_{cb} < \infty$;*

Noncommutative Poisson transforms of c.b. maps

(iii) there exists a k-tuple $\mathbf{V} = (V_1, \dots, V_k)$ of doubly commuting row isometries acting on $\mathcal{K}$ and bounded linear operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ such that

$$F(\mathbf{X}) = (W_1^* \otimes I)\,[C_{\mathbf{X}}(\mathbf{V})^* C_{\mathbf{X}}(\mathbf{V})]\,(W_2 \otimes I),$$

where

$$C_{\mathbf{X}}(\mathbf{V}) := (I \otimes \boldsymbol{\Delta}_{\mathbf{X}}(I)^{1/2}) \prod_{i=1}^{k} (I - V_{i,1} \otimes X_{i,1}^* - \cdots - V_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

Moreover, in this case we can choose μ such that

$$\|\mu\|_{cb} = \sup_{0 \le r < 1} \|\nu_{F_r}\|_{cb}.$$

Noncommutative Poisson transforms of c.p. maps

Corollary

Let $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ be a free k-pluriharmonic function. Then the following statements are equivalent :

(i) *there exists a completely positive linear map $\mu : C^*(\mathbf{R}) \to B(\mathcal{E})$ such that $F = \mathcal{P}\mu$;*

(ii) *the linear maps $\{\nu_{F_r}\}_{r\in[0,1)}$ associate with F are completely positive ;*

(iii) *there exists a k-tuple $\mathbf{V} = (V_1, \ldots, V_k)$ of doubly commuting row isometries acting on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ and a bounded operator $W : \mathcal{E} \to \mathcal{K}$ such that*

$$F(\mathbf{X}) = (W^* \otimes I)\left[C_{\mathbf{X}}(\mathbf{V})^* C_{\mathbf{X}}(\mathbf{V})\right](W \otimes I).$$

Noncommutative Poisson transforms of c.p. maps

- Classical result : A map $u : \mathbb{D}^k \to \mathbb{C}$ is a positive k-harmonic function if and only if there is a finite positive Borel measure on $\mathbb{T}^k$ such that

$$u(z) = \int_{\mathbb{T}^k} P(z, \zeta) d\mu(\zeta), \quad z \in \mathbb{D}^k,$$

  where $P(z, \zeta)$ is the Poisson kernel for the polydisk.

- Open question : Is any positive free k-pluriharmonic function on the regular polyball $\mathbf{B_n}$ the noncommutative Poisson transform of a completely positive linear map $\mu : C^*(\mathbf{R}) \to B(\mathcal{E})$?

Noncommutative Poisson transforms of c.p. maps

- The answer is positive for the unit ball $[B(\mathcal{H})^n]_1$ (when $k = 1$) (P., Adv. Math., 2009) and for the regular polydisk $\mathbf{D}^k(\mathcal{H})$ (when $n_1 = \cdots = n_k = 1$).

Theorem

A map $f : \mathbf{D}^k(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is a positive free k-pluriharmonic function on the regular polydisk if and only if there exists a completely positive linear map $\mu : C^*(M_{z_1}, \ldots, M_{z_k}) \to B(\mathcal{E})$ such that $F = \mathcal{P}\mu$, where $M_{z_1}, \ldots, M_{z_k}$ are the multiplication operators on $H^2(\mathbb{D}^k)$.

Poincaré distance on the open unit disc

- The hyperbolic (Poincaré) distance on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ is defined by

$$\delta_P(z, w) := \frac{1}{2} \ln \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \qquad z, w \in \mathbb{D},$$

where φ_z is the automorphism of $\mathbb{D}$ given by $\varphi_z(w) = \frac{w-z}{1-\bar{z}w}$.

Poincaré distance on the open unit disc

- Basic properties of the Poincaré distance :

1. the Poincaré distance is invariant under the conformal automorphisms of $\mathbb{D}$, i.e.,

$$\delta_P(\varphi(z), \varphi(w)) = \delta_P(z, w), \quad z, w \in \mathbb{D},$$

for all $\varphi \in Aut(\mathbb{D})$;

2. the δ_P-topology induced on the open disc is the usual planar topology ;
3. $(\mathbb{D}, \delta_P)$ is a complete metric space ;
4. any analytic function $f : \mathbb{D} \to \mathbb{D}$ is distance-decreasing, i.e.,

$$\delta_P(f(z), f(w)) \leq \delta_P(z, w), \quad z, w \in \mathbb{D}.$$

Extensions of Poincaré distance

• Bergman introduced an analogue of the Poincaré distance for the open unit ball of $\mathbb{C}^n$,

$$\mathbb{B}_n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \ \|z\|_2 < 1\},$$

defined by

$$\beta_n(z, w) = \frac{1}{2} \ln \frac{1 + \|\psi_z(w)\|_2}{1 - \|\psi_z(w)\|_2}, \qquad z, w \in \mathbb{B}_n,$$

where ψ_z is the involutive automorphism of $\mathbb{B}_n$ that interchanges 0 and z. The Poincaré-Bergman distance has properties similar to those of δ_P.

Extensions of Poincaré distance

• There are several extensions of the Poincaré-Bergman distance to more general domains.

1. The work of R.S. Phillips and L. Harris on infinite-dimensional Cartan domains.
2. The work of Suciu, Foiaş, and Andô-Suciu-Timotin on Harnack type distances between two contractions.
3. The work of P. on hyperbolic geometry on $[B(\mathcal{H})^n]_1$.

Harnack domination

- Preorder relation $\overset{H}{\prec}$ on the closed ball $\mathbf{B_n}(\mathcal{H})^-$.

Definition

If $\mathbf{A}$ and $\mathbf{B}$ are in $\mathbf{B_n}(\mathcal{H})^-$, we say that $\mathbf{A}$ is *Harnack dominated* by $\mathbf{B}$, and denote $\mathbf{A} \overset{H}{\prec} \mathbf{B}$, if there exists $c > 0$ such that

$$F(r\mathbf{A}) \leq c^2 F(r\mathbf{B})$$

for any positive free k-pluriharmonic function F with operator valued coefficients and any $r \in [0, 1)$. When we want to emphasize the constant c, we write $\mathbf{A} \overset{H}{\underset{c}{\prec}} \mathbf{B}$.

Harnack equivalence

Definition

If $\mathbf{A}, \mathbf{B} \in \mathbf{B_n}(\mathcal{H})^-$, we say that $\mathbf{A}$ and $\mathbf{B}$ are *Harnack equivalent* (and denote $\mathbf{A} \overset{H}{\sim} \mathbf{B}$) if there exists $c > 1$ such that

$$\frac{1}{c^2} F(r\mathbf{B}) \leq F(r\mathbf{A}) \leq c^2 F(r\mathbf{B}), \qquad r \in [0,1),$$

for any positive free k-pluriharmonic function $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, where $\mathcal{E}$ is a separable Hilbert space. In this case, we write $\mathbf{A} \underset{c}{\overset{H}{\sim}} \mathbf{B}$.

- The equivalence classes with respect to the equivalence relation $\overset{H}{\sim}$ are called Harnack parts of $\mathbf{B_n}(\mathcal{H})^-$.

Poisson domination

- Recall the *free pluriharmonic Poisson kernel* :

$$\mathcal{P}(\mathbf{R},\mathbf{X}) := \sum_{(\alpha,\beta)\in\Lambda} \mathbf{R}^*_{\tilde{\alpha}}\mathbf{R}_{\tilde{\beta}} \otimes \mathbf{X}_\alpha \mathbf{X}^*_\beta$$

for any $\mathbf{X} \in \mathbf{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology.

- If $\mathbf{A}$ and $\mathbf{B}$ are in $\mathbf{B_n}(\mathcal{H})^-$, we say that $\mathbf{A}$ is *Poisson dominated* by $\mathbf{B}$, and denote $\mathbf{A} \overset{P}{\prec} \mathbf{B}$, if there exists $c > 0$ such that

$$\mathcal{P}(\mathbf{R}, r\mathbf{A}) \leq c^2 \mathcal{P}(\mathbf{R}, r\mathbf{B})$$

for any $r \in [0,1)$. When we want to emphasize the constant c, we write $\mathbf{A} \overset{P}{\underset{c}{\prec}} \mathbf{B}$.

Poisson equivalence

Definition

If $\mathbf{A}, \mathbf{B} \in \mathbf{B_n}(\mathcal{H})^-$, we say that $\mathbf{A}$ and $\mathbf{B}$ are Poisson equivalent (we denote $\mathbf{A} \overset{P}{\sim} \mathbf{B}$) if and only if there exists $c \geq 1$ such that

$$\frac{1}{c^2}\mathcal{P}(\mathbf{R}, r\mathbf{B}) \leq \mathcal{P}(\mathbf{R}, r\mathbf{A}) \leq c^2\mathcal{P}(\mathbf{R}, r\mathbf{B})$$

for any $r \in [0, 1)$.

We also use the notation $\mathbf{A} \underset{c}{\overset{P}{\sim}} \mathbf{B}$ if $\mathbf{A} \underset{c}{\overset{P}{\prec}} \mathbf{B}$ and $\mathbf{B} \underset{c}{\overset{P}{\prec}} \mathbf{A}$.

Harnack inequality

Theorem

Let F be a positive free k-pluriharmonic function on the regular polyball $\mathbf{B_n}$, with operator coefficients in $B(\mathcal{E})$ and let $0 \leq r < 1$. Then

$$F(0)\left(\frac{1-r}{1+r}\right)^k \leq F(\mathbf{X}) \leq F(0)\left(\frac{1+r}{1-r}\right)^k$$

for any $\mathbf{X} \in r\mathbf{B_n}(\mathcal{H})^-$.

Harnack and Poisson equivalence class containing 0

Theorem

Let $\mathbf{A} = (A_1, \ldots, A_k) \in \mathbf{B_n}(\mathcal{H})^-$. *Then the following statements are equivalent.*

1. $\mathbf{A} \overset{H}{\sim} 0$ *;*
2. $r(A_i) < 1$ *for any* $i \in \{1, \ldots, k\}$ *and there exists* $a > 0$ *such that*
$$\mathcal{P}(\mathbf{R}, r\mathbf{A}) \geq aI, \qquad r \in [0, 1);$$
3. $\mathbf{A} \in \mathbf{B_n}(\mathcal{H})$ *;*
4. $\mathbf{A} \overset{P}{\sim} 0$.

Hyperbolic metric on Harnack parts

- Given $\mathbf{A}, \mathbf{B} \in \mathbf{B_n}(\mathcal{H})^-$ in the same Harnack part, i.e. $\mathbf{A} \overset{H}{\sim} \mathbf{B}$, we introduce

$$\omega_H(\mathbf{A}, \mathbf{B}) := \inf\left\{c > 1 : \ \mathbf{A} \underset{c}{\overset{H}{\sim}} \mathbf{B}\right\}.$$

Theorem

Let Δ be a Harnack part of $\mathbf{B_n}(\mathcal{H})^-$ and define $\delta_H : \Delta \times \Delta \to \mathbb{R}^+$ by setting

$$\delta_H(\mathbf{A}, \mathbf{B}) := \ln \omega_H(\mathbf{A}, \mathbf{B}), \quad \mathbf{A}, \mathbf{B} \in \Delta.$$

Then δ_H is a metric on Δ.

Hyperbolic metric on Harnack parts

- Schwarz-Pick lemma for free holomorphic functions on the regular polyball $\mathbf{B_n}$ with operator-valued coefficients, with respect to the hyperbolic metric.

Theorem

Let $\Phi = (\Phi_1, \ldots, \Phi_m) : \mathbf{B_n}(\mathcal{H}) \to [B(\mathcal{H})^m]_1^-$ *be a free holomorphic function on the regular polyball. If* $\mathbf{X}, \mathbf{Y} \in \mathbf{B_n}(\mathcal{H})$*, then* $\Phi(\mathbf{X}) \overset{H}{\sim} \Phi(\mathbf{Y})$ *and*

$$\delta_H(\Phi(\mathbf{X}), \Phi(\mathbf{Y})) \leq \delta_H(\mathbf{X}, \mathbf{Y}),$$

where δ_H *is the hyperbolic metric defined on the Harnack parts of* $[B(\mathcal{H})^m]_1^-$ *and on the polyball* $\mathbf{B_n}(\mathcal{H})$*, respectively.*

Hyperbolic metric on Harnack parts

- The hyperbolic metric is invariant under the group $Aut(\mathbf{B_n})$ of all free holomorphic automorphisms of $\mathbf{B_n}$.

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be in* $\mathbf{B_n}(\mathcal{H})^-$ *such that* $\mathbf{A} \overset{H}{\sim} \mathbf{B}$. *Then*

$$\delta_H(\mathbf{A}, \mathbf{B}) = \delta_H(\mathbf{\Psi}(\mathbf{A}), \mathbf{\Psi}(\mathbf{B})), \qquad \mathbf{\Psi} \in Aut(\mathbf{B_n}).$$

Metric on Poisson parts of the polyball

- Given $\mathbf{A}, \mathbf{B} \in \mathbf{B_n}(\mathcal{H})^-$ in the same Poisson part, i.e. $\mathbf{A} \overset{P}{\sim} \mathbf{B}$, we introduce

$$\omega_{\mathcal{P}}(\mathbf{A}, \mathbf{B}) := \inf \left\{ c > 1 : \ \mathbf{A} \underset{c}{\overset{P}{\sim}} \mathbf{B} \right\}.$$

Theorem

Let Δ be a Poisson part of $\mathbf{B_n}(\mathcal{H})^-$ and define the function $\delta_{\mathcal{P}} : \Delta \times \Delta \to \mathbb{R}^+$ by setting

$$\delta_{\mathcal{P}}(\mathbf{A}, \mathbf{B}) := \ln \omega_{\mathcal{P}}(\mathbf{A}, \mathbf{B}), \quad \mathbf{A}, \mathbf{B} \in \Delta.$$

Then $\delta_{\mathcal{P}}$ is a metric on Δ.

Metric on Poisson parts of the polyball

Theorem

If $\mathbf{A}$ *and* $\mathbf{B}$ *are in the open ball* $\mathbf{B_n}(\mathcal{H})$, *then*

$$\delta_{\mathcal{P}}(\mathbf{A}, \mathbf{B}) = \ln \max \left\{ \left\| C_{\mathbf{A}}(\mathbf{R}) C_{\mathbf{B}}(\mathbf{R})^{-1} \right\|, \left\| C_{\mathbf{B}}(\mathbf{R}) C_{\mathbf{A}}(\mathbf{R})^{-1} \right\| \right\},$$

where

$$C_{\mathbf{X}}(\mathbf{R}) := (I \otimes \mathbf{\Delta_X}(I)^{1/2}) \prod_{i=1}^{k} (I - \mathbf{R}_{i,1} \otimes X_{i,1}^* - \cdots - \mathbf{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1}$$

for any $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})$ *with* $X_i = (X_{i,1}, \ldots, X_{i,n_i})$.

Metric on Poisson parts of the polyball

- Set

$$\mathbf{B_n}(\mathcal{H})_0^- := \left\{ \mathbf{X} \in \mathbf{B_n}(\mathcal{H})^- : \ \mathbf{X} \overset{P}{\prec} 0 \right\}$$

and recall that $\mathbf{B_n}(\mathcal{H}) \subset \mathbf{B_n}(\mathcal{H})_0^-$.

Theorem

Let Δ be a Poisson part of $\mathbf{B_n}(\mathcal{H})_0^-$. Then the following properties hold :

(i) *$\delta_{\mathcal{P}}$ is a complete metric on Δ.*

(ii) *the $\delta_{\mathcal{P}}$-topology and the operator norm topology coincide on the open polyball $\mathbf{B_n}(\mathcal{H})$.*

(iii) *the δ_H-topology is stronger that the $\delta_{\mathcal{P}}$-topology on $\mathbf{B_n}(\mathcal{H})$.*

Positive k-harmonic functions on the regular polydisk

Theorem

Let $F : \mathbf{D}^k(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ be a free k-pluriharmonic function. Then the following statements are equivalent :

(i) *F is positive ;*

(ii) *there exists a completely positive linear map $\mu : C^*(\mathbf{R}) \to B(\mathcal{E})$ such that $F = \mathcal{P}\mu$;*

(iii) *there exists a k-tuple $\mathbf{U} = (U_1, \ldots, U_k)$ of commuting unitaries acting on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ and a bounded operator $W : \mathcal{E} \to \mathcal{K}$ such that*

$$F(\mathbf{X}) = (W^* \otimes I)\,[C_{\mathbf{X}}(\mathbf{U})^* C_{\mathbf{X}}(\mathbf{U})]\,(W \otimes I),$$

Positive k-harmonic functions on the regular polydisk

where

$$C_{\mathbf{X}}(\mathbf{U}) := (I \otimes \boldsymbol{\Delta}_{\mathbf{X}}(I)^{1/2}) \prod_{i=1}^{k} (I - U_i \otimes X_i^*)$$

for any $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{D}^k(\mathcal{H})$.

- The Kobayashi distance for the polydisc $\mathbb{D}^k$ is given by

$$K_{\mathbb{D}^k}(\mathbf{z}, \mathbf{w}) = \frac{1}{2} \ln \frac{1 + \|\psi_{\mathbf{z}}(\mathbf{w})\|_\infty}{1 - \|\psi_{\mathbf{z}}(\mathbf{w})\|_\infty},$$

where $\psi_{\mathbf{z}}$ is the involutive automorphisms of $\mathbb{D}^k$ given by

$$\psi_{\mathbf{z}} = \left(\frac{w_1 - z_1}{1 - \bar{z}_1 w_1}, \ldots, \frac{w_k - z_k}{1 - \bar{z}_k w_k} \right)$$

for any $\mathbf{z} = (z_1, \ldots, z_k)$ and $\mathbf{w} = (w_1, \ldots, w_k)$ in $\mathbb{D}^k$.

Hyperbolic metric on the regular polydisk

Theorem

Let $\mathbf{D}^k(\mathcal{H})$ be the regular polydisk. The following statements hold.

(i) *If $\mathbf{A}, \mathbf{B} \in \mathbf{D}^k(\mathcal{H})^-$, then $\mathbf{A} \overset{H}{\sim} \mathbf{B}$ if and only if $\mathbf{A} \overset{P}{\sim} \mathbf{B}$.*

(ii) *The metrics δ_H and $\delta_{\mathcal{P}}$ coincide on the Harnack parts of $\mathbf{D}^k(\mathcal{H})^-$.*

(iii) *If $\mathbf{A}$ and $\mathbf{B}$ are in $\mathbf{D}^k(\mathcal{H})^-$ and $\mathbf{A} \overset{H}{\sim} \mathbf{B}$, then*

$$\delta_H(\mathbf{A}, \mathbf{B}) = \delta_H(\boldsymbol{\Psi}(\mathbf{A}), \boldsymbol{\Psi}(\mathbf{B})), \qquad \boldsymbol{\Psi} \in Aut(\mathbf{D}^k).$$

Hyperbolic metric on the regular polydisk

(iv) If $\mathbf{A}$ and $\mathbf{B}$ are in $\mathbf{D}^k(\mathcal{H})$, then

$$\delta_H(\mathbf{A},\mathbf{B}) = \ln\max\left\{\left\|C_{\mathbf{A}}(\mathbf{R})C_{\mathbf{B}}(\mathbf{R})^{-1}\right\|, \left\|C_{\mathbf{B}}(\mathbf{R})C_{\mathbf{A}}(\mathbf{R})^{-1}\right\|\right\},$$

where

$$C_{\mathbf{X}}(\mathbf{R}) := (I\otimes \mathbf{\Delta}_{\mathbf{X}}(I)^{1/2})\prod_{i=1}^{k}(I - R_i\otimes X_i^*)$$

for any $\mathbf{X} = (X_1,\dots,X_k) \in \mathbf{D}^k(\mathcal{H})$.

Hyperbolic metric on the regular polydisk

(v) $\delta_H|_{\mathbb{D}^k \times \mathbb{D}^k}$ is equivalent to the Kobayashi distance on the polydisk $\mathbb{D}^k$ and

$$\delta_H(\mathbf{z}, \mathbf{w}) = \frac{1}{2} \ln \frac{\prod_{i=1}^k (1 + |\psi_{z_i}(w_i)|)}{\prod_{i=1}^k (1 - |\psi_{z_i}(w_i)|)}$$

for any $\mathbf{z} = (z_1, \ldots, z_k)$ and $\mathbf{w} = (w_1, \ldots, w_k)$ in $\mathbb{D}^k$, where $\psi_{\mathbf{z}} := (\psi_{z_1}, \ldots, \psi_{z_n})$ is the involutive automorphisms of $\mathbb{D}^k$ such that $\psi_{z_i}(0) = z_i$ and $\psi_{z_i}(z_i) = 0$.

(vi) The hyperbolic metric δ_H is complete on the Harnack parts of $\mathbf{D}^k(\mathcal{H})_0^-$.

(vii) The δ_H-topology coincides with the operator norm topology on the regular polydisk $\mathbf{D}^k(\mathcal{H})$.

Hyperbolic metric on the regular polydisk

Corollary

Let $f = (f_1, \ldots, f_m) : \mathbf{D}^k(\mathcal{H}) \to [B(\mathcal{H})^m]_1$ *be a free holomorphic function on the regular polydisk. If* $\mathbf{X}, \mathbf{Y} \in \mathbf{D}^k(\mathcal{H})$, *then*

$$\delta_H(f(\mathbf{X}), f(\mathbf{Y})) \leq \delta_H(\mathbf{X}, \mathbf{Y}),$$

where δ_H *is the hyperbolic metric. In particular, if* $f(0) = 0$, *then*

$$\frac{1 + \|f(\mathbf{z})\|_2}{1 - \|f(\mathbf{z})\|_2} \leq \prod_{i=1}^{k} \frac{1 + |z_i|}{1 - |z_i|}$$

for any $\mathbf{z} = (z_1, \ldots, z_k)$ *in* $\mathbb{D}^k$.

Herglotz-Riesz representations

- Define the space

$$\mathbf{RH}(\mathbf{B_n}) := \text{span}\left\{\Re f : \ f \in Hol_{\mathcal{E}}(\mathbf{B_n})\right\},$$

where $Hol_{\mathcal{E}}(\mathbf{B_n})$ is the set of all free holomorphic functions in the polyball $\mathbf{B_n}$, with coefficients in $B(\mathcal{E})$.

- If $\varphi \in \mathbf{RH}(\mathbf{B_n})$, we consider the family $\{\nu_{\varphi_r}\}_{r\in[0,1)}$ of linear maps $\nu_{\varphi_r} : \boldsymbol{\mathcal{R}}_{\mathbf{n}}^* \boldsymbol{\mathcal{R}}_{\mathbf{n}} \to B(\mathcal{E})$. Note that $\nu_{\varphi_r}(\mathbf{R}_{\boldsymbol{\alpha}}^* \mathbf{R}_{\boldsymbol{\beta}}) = 0$ if $\mathbf{R}_{\boldsymbol{\alpha}}^* \mathbf{R}_{\boldsymbol{\beta}}$ is different from $\mathbf{R}_{\boldsymbol{\gamma}}$ or $\mathbf{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \mathbf{F}_{\mathbf{n}}^+$.

Herglotz-Riesz representations

- Let $\mu : \mathcal{R}_{\mathbf{n}}^* \mathcal{R}_{\mathbf{n}} \to B(\mathcal{E})$ be a completely positive linear map. The *noncommutative Herglotz-Riesz transform* of μ on the regular polyball is the map $\mathbf{H}\mu : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ defined by

$$(\mathbf{H}\mu)(\mathbf{X}) := \widehat{\mu}\left[2\prod_{i=1}^{k}(I - \mathbf{R}_{i,1}^* \otimes X_{i,1} - \cdots - \mathbf{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right]$$

for $\mathbf{X} := (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})$.

Herglotz-Riesz representations

Theorem

Let f be a free holomophic function from the polyball $\mathbf{B_n}(\mathcal{H})$ to $B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$. Then the following statements are equivalent.

(i) *f is a free holomorphic function with $\Re f \geq 0$ and the linear maps $\{\nu_{\Re f_r}\}_{r\in[0,1)}$ associated with $\Re f$ are completely positive.*

(ii) *The function f admits a Herglotz-Riesz representation*

$$f(\mathbf{X}) = (\mathbf{H}\mu)(\mathbf{X}) + i\Im f(0),$$

where $\mu : C^*(\mathbf{R}) \to B(\mathcal{E})$ is a completely positive linear map with the property that $\mu(\mathbf{R}_\alpha^* \mathbf{R}_\beta) = 0$ if $\mathbf{R}_\alpha^* \mathbf{R}_\beta$ is not equal to $\mathbf{R}_\gamma$ or $\mathbf{R}_\gamma^*$ for some $\gamma \in \mathbf{F}_\mathbf{n}^+$.

Herglotz-Riesz representations

(iii) There exist a k-tuple $\mathbf{V} = (V_1, \ldots, V_k)$ of doubly commuting row isometries on a Hilbert space $\mathcal{K}$, and a bounded linear operator $W : \mathcal{E} \to \mathcal{K}$, such that

$$f(\mathbf{X}) = (W^* \otimes I) \left[2 \prod_{i=1}^{k} (I - V_{i,1}^* \otimes X_{i,1} - \cdots - V_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I \right] \times (W \otimes I) + i \Im f(0)$$

and $W^* \mathbf{V}_\alpha^* \mathbf{V}_\beta W = 0$ if $\mathbf{R}_\alpha^* \mathbf{R}_\beta$ is not equal to $\mathbf{R}_\gamma$ or $\mathbf{R}_\gamma^*$ for some $\gamma \in \mathbf{F}_\mathbf{n}^+$.

Herglotz-Riesz representations

- When $n_1 = \cdots = n_k = 1$, we obtain an operator-valued extension of Korányi-Pukánszky integral representation.

Theorem

If $n_1 = \cdots = n_k = 1$, then the statements in the theorem above are equivalent to

(iv) *The map $f : \mathbf{D}^k(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is a free holomorphic function on the regular polydisk and $\Re f \geq 0$.*

Herglotz-Riesz representations

- Korányi-Pukánzky result :

Theorem

A function $f : \mathbb{D}^k \to \mathbb{C}$ is holomorphic and $\Re f \geq 0$ if and only if it admits a representation

$$f(z) = i\Im f(0) + \int_{\mathbb{T}^k} \left[2 \prod_{j=1}^{k} \frac{1}{1 - z_j \bar{\zeta_j}} - 1 \right] d\mu(\zeta)$$

where μ is a positive measure on $\mathbb{T}^k$ such that, unless $m_j \geq 0$ for any $j \in \{1, \ldots, k\}$ or $m_k \leq 0$ for any $j \in \{1, \ldots, k\}$,

$$\int_{\mathbb{T}^k} \zeta_1^{m_1} \cdots \zeta_k^{m_k} d\mu(\zeta) = 0.$$

THANK YOU

Naimark dilations

- We provide a Naimark type dilation theorem for direct products $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ of unital free semigroups and characterize the positive free k-pluriharmonic functions.
- Let $\mathbf{F}_\mathbf{n}^+ := \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ be the unital semigroup with neutral element $\mathbf{g} := (g_0^1, \ldots, g_0^k)$.
- Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)$, $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k)$, and $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_k)$ be in $\mathbf{F}_\mathbf{n}^+$.

Definition

We say that $K : \mathbf{F}_\mathbf{n}^+ \times \mathbf{F}_\mathbf{n}^+ \to B(\mathcal{E})$ is a *left k-multi-Toeplitz kernel* if $K(\mathbf{g}, \mathbf{g}) = I_\mathcal{E}$ and

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \begin{cases} K(\boldsymbol{\alpha}, \boldsymbol{\beta}) & \text{if } \mathbf{S}_{\boldsymbol{\sigma}}^* \mathbf{S}_{\boldsymbol{\omega}} = \mathbf{S}_{\boldsymbol{\alpha}}^* \mathbf{S}_{\boldsymbol{\beta}} \\ 0 & \text{if } \mathbf{S}_{\boldsymbol{\sigma}}^* \mathbf{S}_{\boldsymbol{\omega}} = 0. \end{cases}$$

Naimark dilations

- We say that $\Gamma : \mathbf{F}_\mathbf{n}^+ \times \mathbf{F}_\mathbf{n}^+ \to B(\mathcal{E})$ is a *right k-multi-Toeplitz kernel* if $\Gamma(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\omega}}) = K(\boldsymbol{\sigma}, \boldsymbol{\omega})$, where $\widetilde{\boldsymbol{\sigma}} = (\widetilde{\sigma_1}, \ldots, \widetilde{\sigma_k})$ and $\widetilde{\sigma}_i := g^i_{j_m} \cdots g^i_{j_1}$ is the reverse of $\sigma_i := g^i_{j_1} \cdots g^i_{j_m}$.

Definition

A map $K : \mathbf{F}_\mathbf{n}^+ \times \mathbf{F}_\mathbf{n}^+ \to B(\mathcal{E})$ has a Naimark dilation if there exists a k-tuple of commuting row isometries $\mathbf{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, on a Hilbert space $\mathcal{K} \supset \mathcal{E}$, i.e. the non-selfadjoint algebra $Alg(V_i)$ commutes with $Alg(V_s)$ for any $i, s \in \{1, \ldots, k\}$ with $i \neq s$, such that

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = P_\mathcal{E} \mathbf{V}^*_{\boldsymbol{\sigma}} \mathbf{V}_{\boldsymbol{\omega}}|_\mathcal{E}, \qquad \boldsymbol{\sigma}, \boldsymbol{\omega} \in \mathbf{F}_\mathbf{n}^+.$$

The dilation is called minimal if $\mathcal{K} = \bigvee_{\boldsymbol{\omega} \in \mathbf{F}_\mathbf{n}^+} \mathbf{V}_{\boldsymbol{\omega}} \mathcal{E}$.

Naimark dilations

Theorem

A map $K : \mathbf{F}_\mathbf{n}^+ \times \mathbf{F}_\mathbf{n}^+ \to B(\mathcal{H})$ is a positive semi-definite left k-multi-Toeplitz kernel on $\mathbf{F}_\mathbf{n}^+$ if and only if it admits a Naimark dilation. In this case, there is a minimal dilation which is uniquely determined up to an isomorphism.

Theorem

A map $\Gamma : \mathbf{F}_\mathbf{n}^+ \times \mathbf{F}_\mathbf{n}^+ \to B(\mathcal{H})$ is a positive semi-definite right k-multi-Toeplitz kernel on $\mathbf{F}_\mathbf{n}^+$ if and only if it admits a Naimark dilation. In this case, there is a minimal dilation which is uniquely determined up to an isomorphism.

Schur type results

- If F is a free k-pluriharmonic function on the polyball $\mathbf{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$, one can associate a right k-multi-Toeplitz kernel Γ_F on $\mathbf{F_n^+}$ in terms of the coefficients of F.
- Schur type result for positive k-pluriharmonic functions in polyballs.

Theorem

Let F be a k-pluriharmonic function on the regular polyball $\mathbf{B_n}$, with coefficients in $B(\mathcal{E})$. Then F is positive on $\mathbf{B_n}$ if and only if the kernel Γ_{F_r} is positive semi-definite for any $r \in [0, 1)$, where F_r stands for the mapping $\mathbf{X} \mapsto F(r\mathbf{X})$.

Schur type results

Definition

A free holomorphic function on the polyball $\mathbf{B_n}$ and with operator-valued coefficients in $B(\mathcal{E})$ has the form

$$f(\mathbf{X}) = \sum_{m_1 \in \mathbb{N}} \cdots \sum_{m_k \in \mathbb{N}} \sum_{\substack{\alpha_i \in \mathbb{F}_{n_i}^+, i \in \{1,\ldots,k\} \\ |\alpha_i| = m_i}} A_{(\alpha_1,\ldots,\alpha_k)} \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k},$$

where $\mathbf{X} = (X_1, \ldots, X_k) \in \mathbf{B_n}(\mathcal{H})$ and the series converge in the operator norm topology.

Schur type results

Corollary

Let $f : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ *be a free holomorphic function. Then the following statements are equivalent.*

(i) $\Re f \geq 0$ *on the polyball* $\mathbf{B_n}$ *;*

(ii) $\Re f(r\mathbf{S}) \geq 0$ *for any* $r \in [0, 1)$ *;*

(iii) *the right* k*-multi Toeplitz kernel* $\Gamma_{\Re f_r}$ *is positive semidefinite for any* $r \in [0, 1)$*.*

Positive k-pluriharmonic functions

Theorem

A map $F : \mathbf{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$, with $F(0) = I$, is a positive free k-pluriharmonic function on the regular polyball if and only if it has the form

$$F(\mathbf{X}) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda} P_{\mathcal{E}} \mathbf{V}^*_{\tilde{\boldsymbol{\alpha}}} \mathbf{V}_{\tilde{\boldsymbol{\beta}}}|_{\mathcal{E}} \otimes \mathbf{X}_{\boldsymbol{\alpha}} \mathbf{X}^*_{\boldsymbol{\beta}},$$

where $\mathbf{V} = (V_1, \ldots, V_k)$ is a k-tuple of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ such that

$$\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda} P_{\mathcal{E}} \mathbf{V}^*_{\tilde{\boldsymbol{\alpha}}} \mathbf{V}_{\tilde{\boldsymbol{\beta}}}|_{\mathcal{E}} \otimes r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \mathbf{S}_{\boldsymbol{\alpha}} \mathbf{S}^*_{\boldsymbol{\beta}} \geq 0, \qquad r \in [0, 1),$$

and the series is convergent in the operator topology.

Positive k-pluriharmonic functions

Definition

A k-tuple $\mathbf{V} = (V_1, \ldots, V_k)$ of commuting row isometries $V_i = (V_{i,1}, \ldots, V_{i,n_i})$ is called *pluriharmonic* if the free k-pluriharmonic Poisson kernel

$$\mathcal{P}(\mathbf{V}, r\mathbf{S}) := \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda} \mathbf{V}^*_{\tilde{\boldsymbol{\alpha}}}\mathbf{V}_{\tilde{\boldsymbol{\beta}}} \otimes r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\mathbf{S}_{\boldsymbol{\alpha}}\mathbf{S}^*_{\boldsymbol{\beta}}$$

is a positive operator for any $r \in [0, 1)$.

- Example : $\mathbf{V} := \mathbf{R} = (\mathbf{R}_1, \ldots, \mathbf{R}_k)$.

Positive k-pluriharmonic functions

Proposition

Let $\mathbf{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, *be a* k*-tuple of commuting row isometries. Then* $\mathbf{V}$ *is pluriharmonic in each of the following particular cases :*

(i) *if* $k = 1$ *and* $n_1 \in \mathbb{N}$ *;*

(ii) *if* $\mathbf{V}$ *is doubly commuting, i.e. the* C^**-algebra* $C^*(V_i)$ *commutes with* $C^*(V_s)$ *if* $i, s \in \{1, \ldots, k\}$ *with* $i \neq s$ *;*

(iii) *if* $n_1 = \cdots = n_k = 1$.

Positive k-pluriharmonic functions

Proposition

Let $\mathbf{V} = (V_1, \ldots, V_k)$ be a pluriharmonic tuple of commuting row isometries on a Hilbert space $\mathcal{K}$ and let $\mathcal{E} \subset \mathcal{K}$ be a subspace. Then the map

$$F(\mathbf{X}) := (P_{\mathcal{E}} \otimes I)\mathcal{P}(\mathbf{V}, \mathbf{X})|_{\mathcal{E}\otimes\mathcal{H}}, \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H})$$

is a positive free k-pluriharmonic function on the polyball $\mathbf{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$, and $F(0) = I$. Moreover, in the particular cases when : $k = 1$ (P., Adv. Math., 2009), or when $n_1 = \cdots = n_k = 1$, each positive free k-pluriharmonic function F with $F(0) = I$ has the form above.

Positive k-pluriharmonic functions

- In particular, we obtain a structure theorem for the positive k-harmonic functions on the regular polydisk $\mathbf{D}^k(\mathcal{H})$, which extends the corresponding classical result on scalar polydisks.

Corollary

A map $F : \mathbf{D}^k(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ is positive free k-pluriharmonic function with $F(0) = I$ if and only if there is a k-tuple of doubly commuting isometries $\mathbf{V} = (V_1, \ldots, V_k)$ on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ such that

$$F(\mathbf{X}) := (P_{\mathcal{E}} \otimes I)\mathcal{P}(\mathbf{V}, \mathbf{X})|_{\mathcal{E}\otimes\mathcal{H}}, \qquad \mathbf{X} \in \mathbf{B_n}(\mathcal{H}).$$