Bergman inner functions and wandering subspaces

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Multivariable Operator Theory at the Technion
On the occasion of Baruch Solel's 65th birthday
June 18-22, 2017

Contractions

Let $T \in L(H)^n$ be a commuting tuple

T is a row contraction if $H^n \xrightarrow{(T_1,\ldots,T_n)} H$ is a contraction

$$\Leftrightarrow 1_H - TT^* = 1_H - \sum_{i=1}^{n} T_i T_i^* \geq 0,$$

or equivalently, if

$$\Leftrightarrow (I - \sigma_T)(1_H) \geq 0,$$

where $\sigma_T : L(H) \to L(H),\ X \mapsto \sum_{i=1}^{n} T_i X T_i^*$.

Standard example: M_z on the functional Hilbert space $H(\mathbb{B})$ with kernel

$$K : \mathbb{B} \times \mathbb{B} \to \mathbb{C}, K(z, w) = \frac{1}{1 - \langle z, w \rangle}$$

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Dilations

For a row contraction $T \in L(H)^n$, define

$$D_{T^*} = (1_H - TT^*)^{1/2}, \mathcal{D} = \overline{D_{T^*} H}$$

Theorem (Müller-Vasilescu, Arveson)

$T \in L(H)^n$ is a row contraction iff

$$T \cong P_{M^\perp}(M_z \oplus U)|_{M^\perp}$$

with $M \in \mathrm{Lat}(M_z \oplus U, H(\mathbb{B}, \mathcal{D}) \oplus K)$ and $U \in L(K)$ is a spherical unitary.

The unitary part $U \in L(K)^n$ does not occur iff T is pure ($\Leftrightarrow C_{\cdot 0}$), that is, if

$$\mathrm{SOT} - \lim_{k \to \infty} \sigma_T^k(1_H) = 0,$$

where again $\sigma_T(X) = \sum_{i=1}^n T_i X T_i^*$.

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m-hypercontractions

What happens if $H(\mathbb{B})$ is replaced by the functional Hilbert space $H_m(\mathbb{B})$ with kernel

$$K_m : \mathbb{B} \times \mathbb{B} \to \mathbb{C}, K_m(z,w) = \frac{1}{(1 - \langle z, w \rangle)^m}?$$

Then even all defect operators

$$\Delta_{M_z}^{(k)} = (I - \sigma_{M_z})^k (1_{H_m(\mathbb{B})}) \geq 0 \quad (k = 0, \ldots, m)$$

are positive.

Definition

$T \in L(H)^n$ is called an *m*-**hypercontraction** if

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In this case: $j_T : (H, T^*) \to (H_m(\mathbb{B}, \mathcal{D}), M_z^*)$,

$j_T x = C(1_H - ZT^*)^{-m} x$ is an isometric intertwiner

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Wandering subspaces

A closed subspace $\mathcal{W} \subset H$ is wandering for $T \in L(H)^n_{com}$ if (Halmos 1961: n = 1)

$$\mathcal{W} \perp T^\alpha \mathcal{W} \quad \forall \alpha \in \mathbb{N}^n \setminus \{0\}.$$

Basic properties:

1. $M \in \mathrm{Lat}(T) \Rightarrow W_T(M) = M \ominus \sum_{i=1}^n T_i M \in \mathrm{Wand}(T)$
2. $M = \bigvee T^\alpha \{x_1, \dots, x_l\} \Rightarrow \dim W_T(M) \leq l$
3. If $\mathcal{W} \in \mathrm{Wand}(T)$, $M = \bigvee T^\alpha \mathcal{W} \Rightarrow \mathcal{W} = W_T(M)$ and

   $T|_M$ is N-cyclic if $N = \dim \mathcal{W} < \infty$.

Theorem (Beurling's theorem)

If $M \in \mathrm{Lat}(M_z, H^2(\mathbb{D}))$, then

(i) $W_{M_z}(M) = \mathbb{C}\theta$ *for some inner function* $\theta \in H^\infty(\mathbb{D})$

(ii) $M = \bigvee_k M_z^k W_{M_z}(M) = \theta H^2(\mathbb{D})$ *(Wandering subspace property)*

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Generalized Bergman spaces

$$H_m(\mathbb{B},\mathcal{D}) = H\left(\frac{1_{\mathcal{D}}}{(1-\langle z,w\rangle)^m}\right)$$
$$= \{f = \sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B},\mathcal{D});\ \|f\|^2 = \sum_{\alpha\in\mathbb{N}^n} \frac{\|f_\alpha\|^2}{\rho_m(\alpha)} < \infty\}$$

Particular cases:

$$H_1(\mathbb{B},\mathcal{D}) = \text{Drury-Arveson space} \overset{n=1}{=} H^2(\mathbb{D},\mathcal{D})$$

$$H_n(\mathbb{B},\mathcal{D}) = \{f \in \mathcal{O}(\mathbb{B},\mathcal{D});\ \|f\|^2 = \sup_{0<r<1} \int_S \|f(r\xi)\|^2 d\xi < \infty\} \text{ Hardy space}$$

$$H_{n+1}(\mathbb{B},\mathcal{D}) = L^2_a(\mathbb{B},\mathcal{D}) = \{f \in \mathcal{O}(\mathbb{B},\mathcal{D});\ \int_{\mathbb{B}} \|f\|^2 dz < \infty\} \text{ Bergman space}$$

$$H_{n+k}(\mathbb{B},\mathcal{D}) = \{f \in \mathcal{O}(\mathbb{B},\mathcal{D});\ \int_{\mathbb{B}} \|f\|^2 (1-|z|^2)^k dz < \infty\}$$

Wandering subspace property and index

The index of $M \in \mathrm{Lat}(T)$ is defined as

$$\mathrm{ind}(M) = \dim W_T(M) = \dim M \ominus \left(\sum_{i=1}^{n} T_i M\right)$$

	Wandering subspace property	$\mathrm{ind}(M) < \infty \forall M \in \mathrm{Lat}(M_z)$
$H^2(\mathbb{D})$	Yes: Beurling	Yes: Beurling
$L^2_a(\mathbb{D})$	Yes: Aleman-Richter-Sundberg	No: Scott Brown, Hedenmalm
$H_m(\mathbb{D})$	Yes $(1 \leq m \leq 3)$: Hedenmalm, Shimorin No $(m \geq 6)$: Hedenmalm	No!

What happens in the multivariable case $n \geq 2$?

Candidate for best possible results: $H_1(\mathbb{B}) = H\left(\frac{1}{1-\langle z,w\rangle}\right)$

- Beurling-Lax-Halmos thm.: McCullough-Trent
- Nevanlinna-Pick interpolation: Ball-Bolotnikov, E.-Putinar
- Dilation theory: Müller-Vasilescu, Arveson, Davidson
- Non-commutative (Fock space) versions: Popescu

However:

- $\exists\, M \in \mathrm{Lat}(M_z, H_1(\mathbb{B}))$ with $\mathrm{ind}(M) = \infty$ (Green-Richter-Sundberg)
- $M_a = \{f \in H_1(\mathbb{B});\ f(a) = 0\} \Rightarrow W_{M_z}(M_a) = \mathbb{C}\theta_a$ for $a \neq 0$

$$\Rightarrow Z(M_a) = \{a\} \neq Z(\theta_a) = Z([W_{M_z}(M_a)]).$$

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Homogeneous wandering subspace property

Suppose that H has an orthogonal decomposition

$$H = \bigoplus_{k=0}^{\infty} H_k$$

and that $T \in L(H)^n_{\mathrm{com}}$ is homogeneous, that is,

$$T_i H_k \subset H_{k+1} \quad (i = 1, \ldots, n, k \geq 0).$$

A closed subspace $M \subset H$ is called homogeneous if

$$M = \bigvee_{k \geq 0} M \cap H_k \quad (\Leftrightarrow P_{H_k} M \subset M \; \forall k)$$

T has the homogeneous wandering subspace property if

$$M = \bigvee_{\alpha \in \mathbb{N}^n} T^{\alpha} W_T(M) \quad \forall M \in \mathrm{Lat}_{\mathrm{hom}}(T)$$

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The homogeneous world is OK

Let $T \in L(H)^n$ be homogeneous.

Theorem

T has the homogeneous wandering subspace property and

$$\mathrm{Wand}_{\mathrm{hom}}(T) \to \mathrm{Lat}_{\mathrm{hom}}(T), \mathcal{W} \mapsto \bigvee_{\alpha \in \mathbb{N}^n} T^\alpha \mathcal{W},$$

$$\mathrm{Lat}_{\mathrm{hom}}(T) \to \mathrm{Wand}_{\mathrm{hom}}(T), M \mapsto W_T(M) = M \ominus (\sum_{i=1}^{n} T_i M)$$

are bijections that are inverse to each other.

Main Ideas: Show that $W_T(H)$ is homogeneous with components

$$W_T(H) \cap H_k = H_k \ominus (\sum_{i=1}^{n} T_i H_{k-1})$$

and show by induction that $H_k \subset [W_T(H)]$ using

$$H_k = (W_T(H) \cap H_k) \oplus \overline{(\sum_{i=1}^{n} T_i H_{k-1})} \subset W_T(H) + [H_{k-1}].$$

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What about the index

Suppose in addition that

$$\dim H_0 < \infty \text{ and } H = \bigvee(T^\alpha H_0; \alpha \in \mathbb{N}^n).$$

Then

1. $H_k = \sum_{|\alpha|=k} T^\alpha H_0 \quad (k \geq 0)$
2. $\tilde{H} = \oplus_{k\geq 0} H_k$ is a finitely generated $\mathbb{C}[z]$-module $(px = p(T)x)$.

Corollary ($\mathcal{W} \in \mathrm{Wand}_{\mathrm{hom}}(T)$, $M \in \mathrm{Lat}_{\mathrm{hom}}(T)$)

- $\dim \mathcal{W} < \infty$
- $M = [W_T(M)]$, $N := \mathrm{ind}(M) = \dim W_T(M) < \infty$ $T|_M$ *N-cyclic*
- *each basis of $W_T(M)$ generates $\tilde{M}$ as a $\mathbb{C}[z]$-module.*

Proof. $\tilde{M} = \oplus_{k\geq 0} M_k$ is finitely generated as a $\mathbb{C}[z]$-submodule $\tilde{M} \subset \tilde{H}$ and

$$M = \bigvee_{\alpha\in\mathbb{N}^n} T^\alpha\{x_1, \ldots, x_\ell\} \text{ for any set of generators } (\Rightarrow \dim W_T(M) \leq \ell).$$

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Applications

Corollary

Let $M \in \mathrm{Lat}_{\mathrm{hom}}(T), \; N = \dim W_T(M)$

(a) $\rho : \tilde{M}/\sum_{i=1}^n T_i\tilde{M} \to M/\overline{\sum_{i=1}^n T_i M} \cong W_T(M), \; [x] \mapsto [x]$ *is an isomorphism.*

(b) $x_i \in M_{k_i}(1 \leq i \leq N)$ *generators for* $\tilde{M} \Rightarrow \forall\, k \in \mathbb{N}$

$$\rho(\{[x_i]; \; k_i = k\}) \text{ is a basis of } W_T(M) \cap H_k.$$

Corollary

$I \subset \mathbb{C}[z_1, \ldots, z_n]$ *homogeneous ideal,* $M = \bar{I} \subset H_m(\mathbb{B})$

(a) $\mathcal{W} = M \ominus \left(\sum_{i=1}^n z_i M\right) = I \ominus \sum_{i=1}^n z_i I \subset I$

(b) *each basis of* $\mathcal{W}$ *is a* **mínimal set of generators** *for* I

(c) *homogeneous generators of degree* $k \triangleq$ *basis of* $\mathcal{W} \cap \mathbb{H}_k \; \forall\, k$.

K_m-inner functions

Hedenmalm'91: $f \in L^2_a(\mathbb{D})$ is Bergman inner if

$$(*) \quad \int_{\mathbb{D}} (|f(z)|^2 - 1) z^k \, dz = 0 \quad \forall\, k \geq 0.$$

Theorem (Hedenmalm '91, Zhu '96)

If f is Bergman inner, then

(a) $H^2(\mathbb{D}) \to L^2_a(\mathbb{D}), \; g \mapsto fg$ *is a contractive multiplier*

(b) $|f(z)|^2 \leq 1/(1-|z|^2) \quad \forall\, z \in \mathbb{D}$

$(*) \quad \Leftrightarrow \quad W_f : \mathbb{D} \to L(\mathbb{C}) \cong \mathbb{C}, \; z \mapsto M_{f(z)}$ satisfies

(i) $\mathbb{C} \to L^2_a(\mathbb{D}), \; \alpha \mapsto W_f \alpha$ is an isometry

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Parametrization of wandering subspaces

Let $\mathcal{E}_*, \mathcal{E}$ be Hilbert spaces, $H(K) \subset \mathcal{O}(\mathbb{B})$ a functional Hilbert space such that

$$M_z \in L(H(K))^n \text{ is a row contraction.}$$

A function $W : \mathbb{B} \to L(\mathcal{E}_*, \mathcal{E})$ is called K-inner if

(i) $\mathcal{E}_* \to H(K, \mathcal{E}), x \mapsto Wx$ is an isometry,

(ii) $(W\mathcal{E}_*) \perp M_z^\alpha(W\mathcal{E}_*) \; \forall \alpha \in \mathbb{N}^n \setminus \{0\}$.

Theorem (Bhattacharjee, Keshari, Sarkar, E.)

(i) *W K-inner $\Rightarrow M_W : H_1(\mathbb{B}, \mathcal{E}_*) \to H(K, \mathcal{E})$ contractive multiplier*

(ii) $\mathrm{Wand}(M_z, H(K, \mathcal{E})) = \{W\mathcal{E}_*;\ W$ *K-inner for some* $\mathcal{E}_*\}$.

Idea for (ii): For $\mathcal{W} \in \mathrm{Wand}(M_z)$, choose a partially isometric multiplier (Sarkar, Ball)

$$M_\theta : H_1(\mathbb{B}, \mathcal{D}) \to H(K, \mathcal{E}) \text{ with } \mathrm{ran}\, M_\theta = [\mathcal{W}].$$

Define $\mathcal{E}_* = \mathcal{D} \cap (\ker M_\theta)^\perp$ and $W(z) = \theta(z)|_{\mathcal{E}_*}$. Then $\mathcal{W} = W(\mathcal{E}_*)$.

Parametrization of wandering subspaces

Let $\mathcal{E}_*, \mathcal{E}$ be Hilbert spaces, $H(K) \subset \mathcal{O}(\mathbb{B})$ a functional Hilbert space such that

$$M_z \in L(H(K))^n \text{ is a row contraction.}$$

A function $W : \mathbb{B} \to L(\mathcal{E}_*, \mathcal{E})$ is called K-inner if

(i) $\mathcal{E}_* \to H(K, \mathcal{E}), x \mapsto Wx$ is an isometry,
(ii) $(W\mathcal{E}_*) \perp M_z^\alpha(W\mathcal{E}_*)\ \forall \alpha \in \mathbb{N}^n \setminus \{0\}$.

Theorem (Bhattacharjee, Keshari, Sarkar, E.)

(i) *W K-inner $\Rightarrow M_W : H_1(\mathbb{B}, \mathcal{E}_*) \to H(K, \mathcal{E})$ contractive multiplier*
(ii) $\mathrm{Wand}(M_z, H(K, \mathcal{E})) = \{W\mathcal{E}_*;\ W\ K\text{-inner for some } \mathcal{E}_*\}$.

Idea for (ii): For $\mathcal{W} \in \mathrm{Wand}(M_z)$, choose a partially isometric multiplier (Sarkar, Ball)

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K_m-inner functions as transfer functions: $n = 1$

Theorem (Olofsson '06/'07)

A function $W : \mathbb{D} \to L(\mathcal{E}_*, \mathcal{E})$ is K_m-inner iff

there exist a pure m-hypercontraction $T \in L(H)$ and a matrix operator

$$\left(\begin{array}{c|c} T^* & B \\ \hline C & D \end{array}\right) \in L(H \oplus \mathcal{E}_*, H \oplus \mathcal{E})$$

with $W(z) = D + zC\left(\sum_{k=1}^{m}(1_H - zT^*)^{-k}\right)B$ *and*

(i) $C^*C = \Delta_T^{(m)}$

(ii) $D^*C + B^*\left(\sum_{k=0}^{m-1}\Delta_T^{(k)}\right)B = 0$

(iii) $D^*D + B^*\left(\sum_{k=0}^{m-1}\Delta_T^{(k)}\right)B = 1_{\mathcal{E}_*}$.

Here $\Delta_T^{(k)} = (1 - \sigma_T)^k(1_H) \geq 0$ are the kth order defect operators of T.

K_m-inner functions as transfer functions: $n \geq 1$

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$$\text{(iv)} \quad \mathrm{Im}(\oplus j)B \subset M_z^* H_m(\mathbb{B}, \mathcal{E}).$$

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$M \in \mathrm{Lat}(M_z, H_m(\mathbb{B})) \Rightarrow \forall\, f \in W_{M_z}(M) = M \ominus \sum_{i=1}^{n} z_i M$

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K_m-inner function W_T of an m-hypercontraction

(1) If T is a pure m-hypercontraction, then $j_T : (H, T^*) \to (H_m(\mathbb{B}, \mathcal{D}), M_z^*)$,

$$j_T(x) = C(1_H - ZT^*)^{-m}x \quad \text{is an isometric intertwiner,}$$

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Construct a K_m-inner function W_T with $W_T\mathcal{E}_* = W_T(M)$ of the right form.

(2) For $W : \mathbb{B} \to L(\mathcal{E}_*, \mathcal{E})$ K_m-inner, apply (1) to

$$T = P_{[\mathcal{W}]^\perp} M_z|_{[\mathcal{W}]^\perp}, \qquad \mathcal{W} = W\mathcal{E}_* \in \operatorname{Wand}(M_z),$$

and show that $W \cong W_T$.

One of the problems: For $n = 1$ the operator

$$L = (M_z^* M_z)^{-1} M_z^* \quad \text{solves Gleason's problem } M_z Lf = f - f(0).$$

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Characteristic functions

For a single pure contraction $T \in L(H)$ (i.e. $m = 1 = n$), the function

$$W_T : \mathbb{D} \to L(\mathcal{D}_T, \mathcal{D}_{T^*}), W_T(z) = D + C(1_H - ZT^*)^{-1}ZB$$

coincides with the Sz.-Nagy-Foias characteristic function θ_T of T.

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Characteristic function: An idea

One can define a purely contractive analytic function $\theta_T : \mathbb{B} \to L(\mathcal{D}_T \oplus M, \mathcal{D})$

$$\theta_T(z) = -\Delta_1(z)(T \oplus 1_M) + \Delta_0(1_H - ZT^*)^{-m} Z(D_T, 0_M)$$

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By abstract results there is an isometry $V : \mathcal{E}_* \to \mathcal{D}_T \oplus M$ such that

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Describe V explicitly, or give a better definition of a characteristic function.

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