Multipartite rational functions

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Multivariable Operator Theory

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Plan

1. Introduction: noncommutative rational functions
2. Multipartite rational functions: construction and universality
3. Amitsur's theorem on multipartite identities
4. Noncommutative function theory perspective

Nc rational expressions

$\mathbb{k}$ a field of characteristic 0, $\mathbf{x} = \{x_1, \ldots, x_g\}$ freely noncommuting letters, $\mathbb{k}\langle\mathbf{x}\rangle$ **the free algebra of nc polynomials**.

$\mathcal{R}_{\mathbb{k}}(\mathbf{x})$ **nc rational expressions** built from $\mathbb{k}\langle\mathbf{x}\rangle$ using
$+,\ \cdot,\ ^{-1},\ (,\ ),$
e.g. $x_2(1+x_1x_2^{-1}(x_1-3))^{-1},\ (x_1x_2)^{-1} - x_2^{-1}x_1^{-1},\ (1-x_1^{-1}x_1)^{-1}.$

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Evaluations of $r \in \mathcal{R}_{\mathbb{k}}(\mathbf{x})$ on tuples of matrices:

- $\mathrm{M}_n(\mathbb{k})^g \dashrightarrow \mathrm{M}_n(\mathbb{k})$ for all $n \in \mathbb{N}$;
- $\operatorname{dom} r \subseteq \bigcup_n \mathrm{M}_n(\mathbb{k})^g$ the **domain** of r;
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Define equivalence relation for nondegenerate expressions: $r_1 \sim r_2$ iff $r_1(X) = r_2(X)$ for all $X \in \operatorname{dom} r_1 \cap \operatorname{dom} r_2$.

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Classes of nondegenerate expressions are called **nc rational functions** and form a skew field $\mathbb{k}\mathopen{(\!\!<}\mathbf{x}\mathclose{>\!\!)}$, the **free skew field**.

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This construction is due to Helton, McCullough, Vinnikov. Others:

- evaluations on ∞-dim skew fields (Amitsur)
- full matrices over $\Bbbk\!<\!\mathbf{x}\!>$ (Cohn)
- Malcev-Neumann series of a free group (Lewin)
- grading on a free Lie algebra (Lichtman)
- unbounded operators associated to a von Neumann algebra (Linnell)

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Here $f = (f_n)_n$, $f_n : \Omega_n \subseteq \mathrm{M}_n(\Bbbk)^g \to \mathrm{M}_n(\Bbbk)$, is a **nc function** if $f_{m+n}(X \oplus Y) = f_m(X) \oplus f_n(Y)$ and $f_n(PXP^{-1}) = Pf_n(X)P^{-1}$.

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If f is a nc function, then

$$f\begin{pmatrix} X & H \\ 0 & Y \end{pmatrix} = \begin{pmatrix} f(X) & \sum_j \Delta_j(f)(X,Y)H_j \\ 0 & f(Y) \end{pmatrix},$$

where Δ_j are (left) **directional nc difference-differential operators**

$$\Delta_j(f)_{m,n} : \Omega_m \times \Omega_n \to \mathrm{Hom}_{\Bbbk}(\Bbbk^{m\times n}, \Bbbk^{m\times n}).$$

(**higher order nc functions**)

Polynomial example

For example, if $f = x_1^2 x_2 x_1$, then the directional nc difference-differential operators of f at $(X_1, X_2; Y_1, Y_2)$ are given by

$$\Delta_1(f)(X_1, X_2; Y_1, Y_2)H = HY_1Y_2Y_1 + X_1HY_2Y_1 + X_1^2X_2H$$

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Hence $\Delta_1, \Delta_2 : \Bbbk\langle\mathbf{x}\rangle \to \Bbbk\langle\mathbf{x}\rangle \otimes \Bbbk\langle\mathbf{y}\rangle$.

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Applying Δ_j further: $\Bbbk\langle\mathbf{x}^{(1)}\rangle \otimes \cdots \otimes \Bbbk\langle\mathbf{x}^{(G)}\rangle$. What are higher order nc rational functions?

Universal skew field of fractions

$\Bbbk\!\!\not<\mathbf{x}\!\!\not>$ is the **universal skew field of fractions** of $\Bbbk\langle\mathbf{x}\rangle$ (Cohn; Amitsur; Kaliuzhnyi-Verbovetskyi, Vinnikov).

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Furthermore, U is a **USFF** of R if for every matrix A over R and a homomorphism $\phi : R \to D$ into a skew field D,

$$\phi(A) \text{ invertible over } D \quad \Rightarrow \quad A \text{ invertible over } U.$$

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This notion is due to Cohn (70s). It is a universal property in the category of skew fields with epimorphisms from R; morphisms are specializations (local homomorphisms) between skew fields.

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Today: $\Bbbk\langle\mathbf{x}^{(1)}\leftrightarrow\cdots\leftrightarrow\mathbf{x}^{(G)}\rangle := \Bbbk\langle\mathbf{x}^{(1)}\rangle \otimes \cdots \otimes \Bbbk\langle\mathbf{x}^{(G)}\rangle$ admits the USFF $\Bbbk(\!\langle\mathbf{x}^{(1)}\leftrightarrow\cdots\leftrightarrow\mathbf{x}^{(G)}\rangle\!)$ for every $G \in \mathbb{N}$.

Notation

For $i = 1, \ldots, G$ let $\mathbf{x}^{(i)} = \{x_1^{(i)}, \ldots, x_{g_i}^{(i)}\}$ be sets of freely noncommuting variables and $\mathbf{x} = \mathbf{x}^{(1)} \cup \cdots \cup \mathbf{x}^{(G)}$.

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Given $r \in \mathcal{R}_{\Bbbk}(\mathbf{x})$ and $X^{(i)} \in \mathrm{M}_{n_i}(\Bbbk)^{g_i}$ we define **mp-evaluation** of r at $X = (X^{(1)}, \ldots, X^{(G)})$ as

$$r^{\mathrm{mp}}(X) :=$$
$$r\left(X^{(1)} \otimes I \otimes \cdots \otimes I, I \otimes X^{(2)} \otimes \cdots \otimes I, \ldots, I \otimes I \otimes \cdots \otimes X^{(G)}\right)$$

in $\mathrm{M}_{n_1 \cdots n_G}(\Bbbk)$, if all nested inverses exist.

Here $\otimes$ denotes Kronecker's product; note that $(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$.

Example

For example, let $\mathbf{x} = \{x_1, x_2\}$, $\mathbf{y} = \{y_1, y_2\}$ and $r = (x_1 + y_2 x_2 x_1 y_1)^{-1} - y_2^{-1}$.

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Then

$$\begin{aligned} r(X;Y) = &\big(X_1 \otimes I + (I \otimes Y_2)(X_2 \otimes I)(X_1 \otimes I)(I \otimes Y_1)\big)^{-1} \\ &- (I \otimes Y_2)^{-1} \end{aligned}$$

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Multipartite rational functions

Given $r \in \mathcal{R}_{\Bbbk}(\mathbf{x})$ let

$$\operatorname{dom}^{\mathrm{mp}} r \subseteq \bigcup_{n_1,\dots,n_G} \mathrm{M}_{n_1}(\Bbbk)^{g_1} \times \cdots \times \mathrm{M}_{n_G}(\Bbbk)^{g_G}$$

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On the set of rational expressions with non-empty mp-domains we define equivalence relation $r_1 \sim r_2$ if and only if $r_1^{\mathrm{mp}}(X) = r_2^{\mathrm{mp}}(X)$ for all $X \in \operatorname{dom}^{\mathrm{mp}} r_1 \cap \operatorname{dom}^{\mathrm{mp}} r_2$. The equivalence class of r is denoted $\mathbf{r}$ and called a **multipartite rational function**.

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The set of multipartite rational functions is denoted $\Bbbk\lbrace\!\!\lbrace \mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)} \rbrace\!\!\rbrace$ and endowed with the natural ring structure.

Theorem

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Basic properties

(1) Let $\mathbf{M} \in \mathrm{M}_d(\Bbbk \langle\!\langle \mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\rangle)$. Then $\mathbf{M}$ is invertible if and only if $\mathbf{M}(X)$ is invertible (as a matrix over $\Bbbk$) for some $X \in \operatorname{dom} \mathbf{M}$.

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(2) Let $\mathbf{r} \in \mathbb{k}\,(\!\!<\mathbf{x}^{(1)}\leftrightarrow\cdots\leftrightarrow\mathbf{x}^{(G)}>\!\!)$ and $Y \in \operatorname{dom}\mathbf{r}$ with $Y^{(1)} \in \mathrm{M}_d(\mathbb{k})^{g_1}$. Then there exists $\mathbf{S} \in \mathrm{M}_d(\mathbb{k}\,(\!\!<\mathbf{x}^{(2)}\leftrightarrow\cdots\leftrightarrow\mathbf{x}^{(G)}>\!\!))$ such that

$$\mathbf{r}(Y^{(1)}, X) = \mathbf{S}(X)$$

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(4) The centralizer of $\mathbb{k}\langle\!\langle \mathbf{x}^{(1)} \rangle\!\rangle$ in $\mathbb{k}\langle\!\langle \mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\rangle$ equals $\mathbb{k}\langle\!\langle \mathbf{x}^{(2)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\rangle$ if $|\mathbf{x}_1| > 1$.

Auxiliary result

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Proposition

Let M be a $d \times d$ matrix over $D \otimes \mathbb{k}{<}\mathbf{x}{>}$. Then M is invertible over the USFF of $D \otimes \mathbb{k}{<}\mathbf{x}{>}$ if and only if $M(X) \in \mathrm{M}_d(D \otimes \mathrm{M}_n(\mathbb{k})) \cong \mathrm{M}_{dn}(D)$ is invertible for some $X \in \mathrm{M}_n(\mathbb{k})^g$.

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Ingredients: Cohn's theory of USFFs, PI theory, skew field constructions and power series expansions.

Universality

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Corollary

Let $r \in \mathcal{R}_{\Bbbk}(\mathbf{x})$. TFAE:

(i) *$r^{\mathrm{mp}}(X) = 0$ for all $X \in \operatorname{dom}^{\mathrm{mp}} r$;*

(ii) *$r(X) = 0$ for all $X \in \operatorname{dom} r$ such that $[X_{j_1}^{(i_1)}, X_{j_2}^{(i_2)}] = 0$ for $i_1 \neq i_2$;*

(iii) *for every skew field D, $r(a) \in \{0, \mathit{undef}\}$ for every tuple $a \in D^{g_1+\cdots+g_G}$ such that $[a_{j_1}^{(i_1)}, a_{j_2}^{(i_2)}] = 0$.*

Sketch of the proof

Let M be a $d \times d$ matrix over $\mathbb{k}{<}\mathbf{x}^{(1)}\leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}{>}$ and let $\phi : \mathbb{k}{<}\mathbf{x}^{(1)}\leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}{>} \to D$ be a homomorphism into a skew field D such that $\phi(M)$ is invertible over M.

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3. proposition: $M(X^{(1)}, a^{(2)}, \dots) \in \mathrm{M}_{dn_1}(D)$ invertible for some $X \in \mathrm{M}_{n_1}(\Bbbk)^{g_1}$

Sketch of the proof

Let M be a $d \times d$ matrix over $\Bbbk{<}\mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}{>}$ and let $\phi : \Bbbk{<}\mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}{>} \to D$ be a homomorphism into a skew field D such that $\phi(M)$ is invertible over M.

1. Write $a_j^{(i)} = \phi(x_j^{(i)})$; $M(a^{(1)}, a^{(2)}, \dots)$ invertible over D
2. $D \otimes \Bbbk{<}\mathbf{x}^{(1)}{>}$ fir: $M(x^{(1)}, a^{(2)}, \dots)$ invertible over the USFF of $D \otimes \Bbbk{<}\mathbf{x}^{(1)}{>}$
3. proposition: $M(X^{(1)}, a^{(2)}, \dots) \in \mathrm{M}_{dn_1}(D)$ invertible for some $X \in \mathrm{M}_{n_1}(\Bbbk)^{g_1}$
4. induction: $N = M(X^{(1)}, x^{(2)}, \dots)$ invertible over $\Bbbk{<\!(}\mathbf{x}^{(2)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}{)\!>}$

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4. induction: $N = M(X^{(1)}, x^{(2)}, \dots)$ invertible over $\Bbbk\langle\!\!{<}\mathbf{x}^{(2)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}{>}\!\!\rangle$
5. basic property: $N(X^{(2)}, \dots)$ invertible for some $X^{(i)} \in \mathrm{M}_{n_i}(\Bbbk)^{g_i}$

Sketch of the proof

Let M be a $d \times d$ matrix over $\Bbbk<\mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}>$ and let $\phi : \Bbbk<\mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}> \to D$ be a homomorphism into a skew field D such that $\phi(M)$ is invertible over M.

1. Write $a_j^{(i)} = \phi(x_j^{(i)})$; $M(a^{(1)}, a^{(2)}, \dots)$ invertible over D
2. $D \otimes \Bbbk<\mathbf{x}^{(1)}>$ fir: $M(x^{(1)}, a^{(2)}, \dots)$ invertible over the USFF of $D \otimes \Bbbk<\mathbf{x}^{(1)}>$
3. proposition: $M(X^{(1)}, a^{(2)}, \dots) \in \mathrm{M}_{dn_1}(D)$ invertible for some $X \in \mathrm{M}_{n_1}(\Bbbk)^{g_1}$
4. induction: $N = M(X^{(1)}, x^{(2)}, \dots)$ invertible over $\Bbbk(\!<\mathbf{x}^{(2)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}>\!)$
5. basic property: $N(X^{(2)}, \dots)$ invertible for some $X^{(i)} \in \mathrm{M}_{n_i}(\Bbbk)^{g_i}$
6. $M(X^{(1)}, X^{(2)}, \dots)$ invertible, so M invertible over $\Bbbk(\!<\mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)}>\!)$

Higher order nc rational functions

Let $\mathbf{r} \in \Bbbk\langle\!\langle \mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\rangle$. Then

1. $\mathbf{r}$ respects direct sums in the first factor and up to canonical shuffle in other factors; ($A \otimes B \sim B \otimes A$)
2. $\mathbf{r}$ respects similarities in every factor.

Hence $\mathbf{r}$ is a nc function of order $G - 1$.

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Directional nc difference-differential operators

$$\Delta_j^{(i)} \colon \Bbbk\langle\!\langle \mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\rangle \to \Bbbk\langle\!\langle \mathbf{x}^{(1)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}'^{(i)} \leftrightarrow \mathbf{x}^{(i)} \leftrightarrow \cdots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\rangle$$

satisfy the usual properties.

Higher order nc rational functions cont'd

Furthermore, diagrams like

$$\begin{tikzcd}
\mathbb{k}\langle\!\!\!\langle \mathbf{x}^{(1)} \cup \mathbf{x}^{(2)} \leftrightarrow \dots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\!\!\rangle \arrow[r, dashed] \arrow[d, "\Delta_j^{(1)}"] & \mathbb{k}\langle\!\!\!\langle \mathbf{x}^{(1)} \leftrightarrow \dots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\!\!\rangle \arrow[d, "\Delta_j^{(1)}"] \\
\mathbb{k}\langle\!\!\!\langle \mathbf{x}'^{(1)} \cup \mathbf{x}'^{(2)} \leftrightarrow \mathbf{x}^{(1)} \cup \mathbf{x}^{(2)} \leftrightarrow \dots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\!\!\rangle \arrow[r, dashed] & \mathbb{k}\langle\!\!\!\langle \mathbf{x}'^{(1)} \leftrightarrow \mathbf{x}^{(1)} \leftrightarrow \dots \leftrightarrow \mathbf{x}^{(G)} \rangle\!\!\!\rangle
\end{tikzcd}$$

commute, where $\dashrightarrow$ are specializations (local homomorphisms) between skew fields.

Thank you,

and happy birthday!