Approximation of Groupoids

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Topology: Terminology and Notation

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Star refinement and uniform structure I

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Say that $\mathcal{U}$ star refines a cover $\mathcal{V}$ (denoted $\mathcal{U} \leq \mathcal{V}$) if, for any $U \in \mathcal{U}$, $\text{star}(U, \mathcal{U})$ is contained in some element of $\mathcal{V}$.
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A paracompact space $X$ has a uniform structure. If $\mathcal{U}$ is an open cover of $X$ then one can find an open cover $\mathcal{V}$ which is a star refinement of $\mathcal{U}$. 
Uniform structure and inverse limits I

An increasing sequence of covers ordered by reverse star refinement is called a **normal sequence**.
Uniform structure and inverse limits I

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If $\Lambda = \{ \mathcal{U}_n \}$ is a normal sequence for $X$, then one can define a pseudo-metric $d$ on $X$:

- For $x, y$ in $X$, let $n(x, y)$ be the largest $n$ such that there exists $U \in \mathcal{U}_n$ containing both $x$ and $y$.
- Let $\rho(x, y) = 2^{-n(x, y)}$, with the understanding that $2^{-\infty} = 0$.
- Let $d(x, y) = \inf \sum_{i=1}^{n} \rho(x_i, x_{i+1})$ where $x_1 = x$, $x_n = y$ and $x_i \in X$. 
On the previous slide we explained how to get a pseudo-metric $d$ from a normal sequence of covers on $X$.

Let $X_\lambda$ be the quotient of $X$ obtained from the equivalence relation $x \sim y$ if and only if $d(x, y) = 0$. The resulting space $X_\lambda$ is a metric space, and $X \to X_\lambda$ is continuous.
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**WARNING:** We will refer to $X_\lambda$ as a quotient of $X$; however, we warn that the topology of $X_\lambda$ is not the canonical quotient topology induced by the original topology on $X$ and the equivalence relation, but is instead determined by the choice of covers $\{ U_n \}$ (i.e. a possibly weaker topology).
Uniform structure and inverse limits III

On the previous slide we explained how to get a metrizable quotient $X_\lambda$ from a topological space $X$ equipped with a normal sequence of covers.

Let $\mathcal{U}$ and $\mathcal{V}$ be two normal sequences for $X$. Say that $\mathcal{V}$ cofinally refines $\mathcal{U}$ if for every $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that $\mathcal{V}_{k(n)} \leq \mathcal{U}_n$. 
Uniform structure and inverse limits III

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Let $\hat{X}_\alpha = \langle X, \mathcal{U} \rangle$ and $\hat{X}_\beta = \langle X, \mathcal{V} \rangle$ (meaning $X$ equipped with the pseudo-metric resulting from $\mathcal{U}$ and $\mathcal{V}$ respectively). If $\mathcal{V}$ cofinally refines $\mathcal{U}$ then:

$\hat{X}_\alpha \leftarrow X \rightarrow \hat{X}_\beta \leftarrow \hat{X}_\alpha$

$\phi = \text{id}$

where the map $\phi : \hat{X}_\beta \rightarrow \hat{X}_\alpha$ is uniformly continuous. If we then define $X_\alpha$ and $X_\beta$ to be the corresponding quotient spaces, it should be clear that $\phi$ then induces a map $X_\beta \rightarrow X_\alpha$. 
Uniform structure and inverse limits IV

Suppose additionally that \( X \) is Lindelöf – that is, that every open cover has a countable subcover. This assumption ensures that we can take each cover in the normal sequence to be countable, and thus that the space \( X_\lambda \) is second countable.
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We also note that for any open cover \( \mathcal{U} \) of \( X \) we can get a normal sequence \( \{ \mathcal{U}_n \} \) for \( X \) such that \( \mathcal{U}_1 = \mathcal{U} \).
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Considering all sequences of covers ordered by cofinal refinement we get:

**Theorem**

*If $X$ is a locally compact and Lindelöf space then it is the inverse limit of a system $\{X_\varphi, p_\psi^\varphi : X_\varphi \to X_\psi\}_{\varphi \in \Lambda}$ where each $X_\varphi$ is second countable and locally compact, and all the connecting maps are proper.*
Groupoids: Definition

One can think of a groupoid $G$ as a set (of arrows), with two operations:

- a partial multiplication $(g, h) \mapsto gh$ defined on a subset $G^{(2)} \subset G \times G$
- an inverse $g \mapsto g^{-1}$ defined on all of $G$. 

The set $G^{(2)}$ is called the set of composable arrows. One defines $s, t: G \to G$ by $s(g) = gg^{-1}$ and $t(g) = g^{-1}g$ (the source and target of each arrow). The following rules must be satisfied:

- the inverse of $g^{-1}$ is $g$.
- $s(g)$ and $t(g)$ act as identity elements for arrows with which they are composable.

The image of the source map (or, equivalently, the target map) is a subset of $G$ denoted $G^{(0)}$, referred to as the unit space of the groupoid.
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Topological groupoids

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Of course, this implies the source and target maps are also continuous.
Examples: Topological Groupoids

- with discrete topology
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- transformation groupoid - starting from $G$ a topological group, $X$ a topological space, and $\alpha : G \curvearrowright X$ a continuous action of $G$ on $X$ by homeomorphisms. The groupoid is basically $X \times G$:
for $x \in G^{(0)}$ write:

- $G^x$ - the set of arrows whose \textit{target} is $x$
- $G_x$ - the set of arrows whose \textit{source} is $x$
Groupoid Notation and Terminology

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Terminology: we say $\mathcal{G}$ is

- **open** if the source and target maps are open maps.
- **étale** if the source and target maps are local homeomorphisms.
- **transitive** if for any $x, y \in \mathcal{G}^{(0)}$ there exists $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$. 

Approximation for Groupoids I

Lemma

Let $G$ be an open Lindelöf groupoid, and $\{K_n\}$ an exhaustion of $G$ by compact sets. There exists a sequence of countable and locally finite open coverings $\{U_n\}_{n \geq 0}$ of $G$ such that for all $n \geq 0$:

1. each set in $U_n$ is pre-compact.
2. $U_{n+1}^0 \leq s(\{K_n \cap U : U \in U_n^1\}), t(\{K_n \cap U : U \in U_n^1\}) \leq U_n^0$.
3. $m(U_{n+1}^1_{|K_n}, U_{n+1}^1_{|K_n}) \leq U_n^1$.
4. $(U_{n+1}^1)^{-1} \leq U_n^1$. 
Approximation for Groupoids II

If we have a normal sequence of covers \( \{ \mathcal{U}_n \} \) for \( G \) satisfying the conditions of the previous slide, then we can form the quotient \( G_\alpha \) (in the same way as described for a general topological space).
Approximation for Groupoids II

If we have a normal sequence of covers \( \{ U_n \} \) for \( G \) satisfying the conditions of the previous slide, then we can form the quotient \( G_\alpha \) (in the same way as described for a general topological space).

The extra conditions we impose on the normal sequence mean that we can define:

- \( s([g]) = [s(g)] \) (where \( g \in G \) is a representative of \( [g] \in G_\alpha \))
- \( [g], [h] \in G_\alpha^{(2)} \) are composable if there exists \( g' \in [g] \) and \( h' \in [h] \) with \( s(h') = t(g') \), in which case \( m([g], [h]) = [g'h'] \)
- \( [g]^{-1} = [g^{-1}] \)

and these operations are well-defined and continuous in \( G_\alpha \).
Haar system of measures

Let $\mathcal{G}$ be a topological groupoid. A **Haar system of measures on** $\mathcal{G}$ is a collection $\{\mu_x : x \in \mathcal{G}^{(0)}\}$ of positive Radon measures on $\mathcal{G}$ such that:

1. $\mu_x$ is supported on $\mathcal{G}^x$
2. for fixed $f \in C_c(\mathcal{G})$, $x \mapsto \int_{\mathcal{G}^{(0)}} f(y) d\mu^x(y)$ is continuous on $\mathcal{G}^{(0)}$
3. for all $g \in \mathcal{G}^{(1)}$ and $f \in C_c(\mathcal{G})$,

$$
\int_{\mathcal{G}^{t(g)}} f(y) d\mu^{t(g)}(y) = \int_{\mathcal{G}^{s(g)}} f(gy) d\mu^{s(g)}(y).
$$

The last condition is the groupoid equivalent of 'left invariance for Haar measure' in the case of groups. The second condition is a continuity condition for the choice of measures, and is needed in order for the convolution product to work (we will discuss this later).
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Having a Haar system on $G$ enables one to construct the groupoid $C^*$-algebra (as explained later).
Approximation of Haar system

We modify the construction of the normal sequence $\mathcal{U}_n$ by the addition of a partition of unity and the following condition:

Fix $\{f^n_\omega : \omega \in \Lambda_n\}$ a finite partition of unity of $K_n$ whose carriers refine $\mathcal{U}_n$. Let $(\lambda_\omega)_\omega \subset \mathbb{C}$ be any sequence with $|\lambda_\omega| < n$. For each element $U \in \mathcal{U}_{n+1}$ and for each $x, y \in s(U)$ we have

$$\left| \int_G \left( \sum_\omega \lambda_\omega f^n_\omega \right) \ d\mu^x - \int_G \left( \sum_\omega \lambda_\omega f^n_\omega \right) \ d\mu^y \right| < \frac{1}{n}.$$
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Let $q : G \to G_\alpha$. It follows that for each $f \in C_c(G_\alpha)$

$$x \sim y \text{ for } x, y \in G^{(0)} \Rightarrow \int_{G^x} (f \circ q) d\mu^x = \int_{G^y} (f \circ q) d\mu^y,$$

allowing us to define a Haar system of measures on $G_\alpha$ based on the Haar system of measures on $G$. 
Assume \( \{ \mu^x \} \) is a Haar system on \( G \).
Equip \( C_c(G) \) with a convolution product and an involution operation. One can then complete the resulting algebra to a \( C^* \)-algebra (in fact, there is a reduced \( C^* \)-algebra and a full \( C^* \)-algebra).
Groupoid $C^*$-algebra I

Assume $\{\mu^x\}$ is a Haar system on $G$.

Equip $\mathcal{C}_c(G)$ with a convolution product and an involution operation. One can then complete the resulting algebra to a $C^*$-algebra (in fact, there is a reduced $C^*$-algebra and a full $C^*$-algebra).

For $\varphi, \psi \in \mathcal{C}_c(G)$ define:

convolution: $(\varphi \ast \psi)(g) = \int \varphi(gh)\psi(h^{-1})d\mu^g(h)$

involution: $\varphi^*(g) = \varphi(g^{-1})$

We omit the description of the reduced and full $C^*$-algebra completion.
Groupoid $C^*$-algebra II

With the construction described so far, $C_c(G_\alpha) \hookrightarrow C_c(G)$ (as a $^*$-algebra embedding).
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This is an easy computational check:

Denote by $q$ the map $G \to G_\alpha$ which takes $g$ to $[g]$. The embedding is given by $\tilde{q}(\varphi) = (\varphi \circ q)$ for $\varphi \in C_c(G_\alpha)$:

\[
\tilde{q}(\varphi \ast \psi)(g) = (\varphi \ast \psi)(q(g)) = \int_{G_\alpha} \varphi(q(g)q(h))\psi(q(h)^{-1}) d\mu^{s(q(g))}(q(h))
\]

\[
= (\tilde{q}(\varphi) \ast \tilde{q}(\psi))(g)
\]

\[
\tilde{q}(\varphi^*)(g) = (\varphi^* \circ q)(g) = \overline{\varphi(q(g)^{-1})} = \overline{\tilde{q}(\varphi)(g^{-1})} = (\tilde{q}(\varphi))^*(g),
\]

where $\varphi, \psi \in C_c(G_\alpha)$ and $g \in G$. We used the fact that $q : G \to G_\alpha$ respects the groupoid operations and is onto.
2-Cocycles and twisted convolution algebra

A 2-cocycle for $\mathcal{G}$ is a map $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ such that

- $\sigma(g, h)\sigma(gh, k) = \sigma(g, hk)\sigma(gh, k)$ for all $(g, h), (h, k) \in \mathcal{G}^{(2)}$, and
- $\sigma(g, s(g)) = 1 = \sigma(t(g), g)$ for all $g \in \mathcal{G}$.

Such a cocycle allows us to construct twisted groupoid $C^*$-algebras (modify the convolution product).
2-Cocycles and twisted convolution algebra

A 2-cocycle for $G$ is a map $\sigma : G^{(2)} \to \mathbb{T}$ such that

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A cocycle on $G$ can be pushed to a cocycle on $G_\alpha$ by again modifying the normal sequence construction to add the following condition:

*Choose a normal sequence $\{V_n\}$ for $\mathbb{T}$, where $V_n$ is a finite cover by $\frac{1}{2^n}$-balls, and also ask that the sequence $\{U_n\}$ satisfies:

$$\sigma(U_n|_{K_n}, U_n|_{K_n}) \leq V_n$$

This ensures that we can define $\sigma([g], [h]) = \sigma(g, h)$ for $g, h \in G$ representatives of $[g], [h] \in G_\alpha$. Similarly to the previous slide, $C_c(G_\alpha, \sigma) \hookrightarrow C_c(G, \sigma)$ (as a $*$-algebra embedding).
Disintegration Theorem

The version of Renault’s theorem mentioned below omits the mention of a 2-cocycle for $\mathcal{G}$ in order to simplify slightly the presentation.

**Theorem (Renault’s Disintegration Theorem)**

Let $\mathcal{G}$ be a second countable locally compact groupoid endowed with a Haar system of measures. Every nondegenerate representation of the *-algebra $C_c(\mathcal{G})$ on a separable Hilbert space is the integrated form of a representation of $\mathcal{G}$ on a bundle of Hilbert spaces.
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Our goal is to use the approximation by second countable groupoids described in the earlier part of the talk to bootstrap this result to $\sigma$-compact groupoids.
Hilbert bundles

A Borel bundle of Hilbert spaces over a space $X$ is a Borel space $Z = X \ast \mathcal{H} = \{(x, v) : x \in X \text{ and } v \in \mathcal{H}_x\}$ such that

1. the projection $p : Z \to X$ is measurable, and each fiber $\mathcal{H}_x$ is a Hilbert space

along with measurable sections $\{s_\alpha\}_{\alpha \in A}$ to $p$ such that for each $x \in X$ the span of the set $\{s_\alpha(x) : \alpha \in A\}$ is dense in $p^{-1}(x)$ and satisfying the following properties

1. for each $\alpha \in A$ the map $(x, v) \to \langle s_\alpha(x), v \rangle$ is measurable on $Z$.
2. for each $\alpha, \beta \in A$ the map $x \to \langle s_\alpha(x), s_\beta(x) \rangle_{\mathcal{H}_x}$ is measurable on $X$.
3. the functions $(x, v) \to \langle s_\alpha(x), v \rangle$ separate the points of $Z$.

If $A = \mathbb{N}$ then we say that the bundle is separable.
Representations of Groupoids

We represent groupoids on a bundle of Hilbert spaces.
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Isomorphism groupoid: \( \text{Iso}(G^{(0)} \ast \mathcal{H}) := \{(x, U, y) : U : \mathcal{H}_x \to \mathcal{H}_y \text{ is unitary}\} \).
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If $G$ is a groupoid equipped with a Haar system of measures, a **unitary representation** of $G$ is a triple $(G^{(0)} \ast \mathcal{H}, L, \nu)$ where

- $G^{(0)} \ast \mathcal{H}$ is a Borel Hilbert bundle over $G^{(0)}$
- $L : G \rightarrow Iso(G^{(0)} \ast \mathcal{H})$ is such that $L(g) = (t(g), L_g, s(g))$
- $\nu$ is a quasi-invariant measure on $G^{(0)}$;
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- \( \nu \) is a quasi-invariant measure on \( G^{(0)} \);

additionally define Borel sections and square-integrable sections

\[
B(G^{(0)} \ast \mathcal{H}) := \{ f : G^{(0)} \rightarrow G^{(0)} \ast \mathcal{H} : x \mapsto \langle f(x), f_\alpha(x) \rangle \text{ is Borel for all } \alpha \} \\
L^2(G^{(0)} \ast \mathcal{H}, \nu) = \{ f \in B(G^{(0)} \ast \mathcal{H}) : x \mapsto \| f(x) \|^2 \text{ is integrable on } G^{(0)} \},
\]
Representations of Groupoids

We represent groupoids on a bundle of Hilbert spaces.

Isomorphism groupoid: \( Iso(\mathcal{G}^{(0)} \ast \mathcal{H}) := \{(x, U, y) : U : \mathcal{H}_x \to \mathcal{H}_y \text{ is unitary}\} \).

If \( \mathcal{G} \) is a groupoid equipped with a Haar system of measures, a **unitary representation** of \( \mathcal{G} \) is a triple \((\mathcal{G}^{(0)} \ast \mathcal{H}, L, \nu)\) where

- \( \mathcal{G}^{(0)} \ast \mathcal{H} \) is a Borel Hilbert bundle over \( \mathcal{G}^{(0)} \)
- \( L : \mathcal{G} \to Iso(\mathcal{G}^{(0)} \ast \mathcal{H}) \) is such that \( L(g) = (t(g), L_g, s(g)) \)
- \( \nu \) is a quasi-invariant measure on \( \mathcal{G}^{(0)} \);

additionally define Borel sections and square-integrable sections
\[
B(\mathcal{G}^{(0)} \ast \mathcal{H}) := \left\{ f : \mathcal{G}^{(0)} \to \mathcal{G}^{(0)} \ast \mathcal{H} : x \mapsto \langle f(x), f_\alpha(x) \rangle \text{ is Borel for all } \alpha \right\}
\]
\[
L^2(\mathcal{G}^{(0)} \ast \mathcal{H}, \nu) = \left\{ f \in B(\mathcal{G}^{(0)} \ast \mathcal{H}) : x \mapsto \|f(x)\|^2 \text{ is integrable on } \mathcal{G}^{(0)} \right\},
\]
and impose the condition that \( g \mapsto \langle L_g h(s(g)), k(t(g)) \rangle \) should be \( \nu \)-measurable for all \( h, k \in L^2(\mathcal{G}^{(0)} \ast \mathcal{H}, \nu) \).
Integrated form of a representation

Suppose $G$ is a groupoid equipped with a Haar system of measures $\{\mu^x\}$. Suppose moreover that $(G^{(0)} \ast \mathcal{H}, L, \nu)$ is a unitary representation of $G$. 

Renault's disintegration theorem states that if $G$ is second countable then all representations of $C^c(G)$ are of this type.
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There is a representation of $C_c(\mathcal{G})$, called the integrated form of $(\mathcal{G}^{(0)} \ast \mathcal{H}, L, \nu)$, denoted by $L$ through a standard abuse of notation, defined such that

$$\langle L(\varphi)h,k \rangle = \int_{\mathcal{G}^{(0)}} \int_{\mathcal{G}} \varphi(g) \langle L_g(h(s(g))),k(t(g)) \rangle \Delta(g)^{-1/2} d\mu^x(g) d\nu(x),$$

where $\varphi \in C_c(\mathcal{G})$, $h,k \in L^2(\mathcal{G}^{(0)} \ast \mathcal{H}, \nu)$ and $\Delta$ is the modular function of $\nu$. 
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Disintegration Theorem for Lindelöf groupoids

We want to extend Renault’s result to Lindelöf groupoids:

**Theorem**

Let $G$ be a Lindelöf locally compact groupoid endowed with a Haar system of measures. Every nondegenerate representation of the $\ast$-algebra $C_c(G)$ on a Hilbert space is the integrated form of a representation of $G$ on a bundle of Hilbert spaces.
Some obvious comments about the quotient construction:

- If $\mathcal{G}$ is transitive, so is $\mathcal{G}_\alpha$.
- If $\mathcal{G}$ is étale, it is easy to ensure the quotient groupoids $\mathcal{G}_\alpha$ are also étale.
Comments, and avenues for future investigation

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Some questions:

- Is it true that if $\mathcal{G}$ is étale then finite dynamic asymptotic dimension is preserved by the construction?
- If $\mathcal{G}$ is equipped with a Fell bundle, is there a good way to associate a Fell bundle to $\mathcal{G}_\alpha$?
- Are there results for second countable groupoids that, using the ideas / constructions presented, can be extended to Lindelöf groupoids?
...The End

Thank you for your attention.