Approximation of Groupoids

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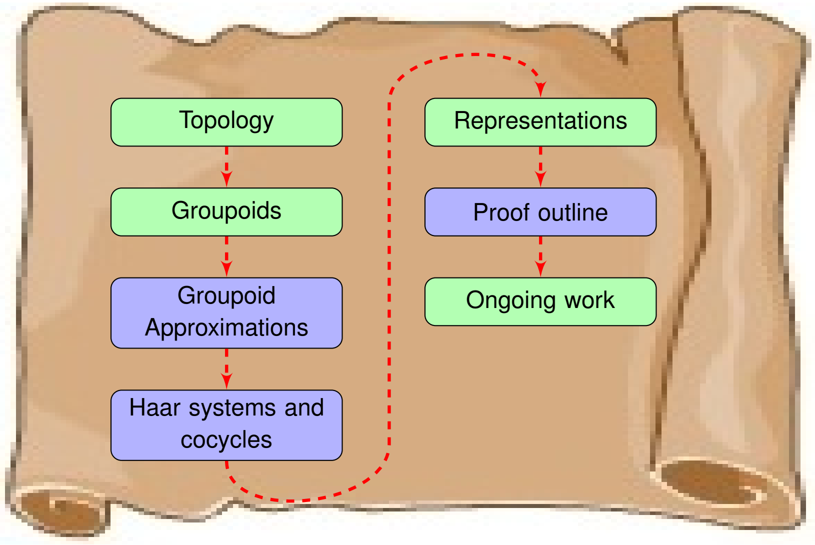
Road map


Topology
Groupoids
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Haar systems and cocycles
Representations
Proof outline
Ongoing work

Topology: Terminology and Notation

X - topological space; $\mathcal{U}$, $\mathcal{V}$ open covers of X

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Paracompactness allows one to endow X with a uniform structure, and hence to write X as an inverse limit of metrizable spaces, as we will explain in just a moment.

$\rightarrow$

Star refinement and uniform structure I

Given $U \in \mathcal{U}$ the star of U against the cover $\mathcal{U}$ is the union of all the sets in $\mathcal{U}$ that intersect U.

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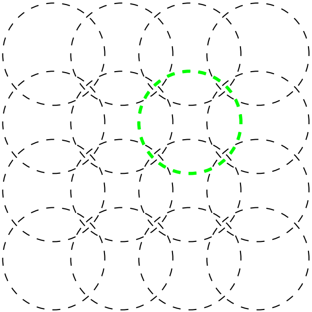
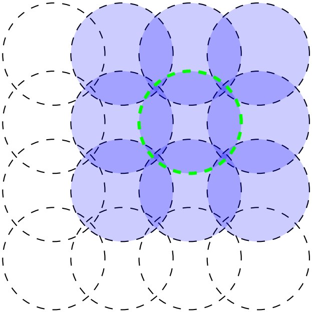


Set U in $\mathcal{U}$.



$star(U, \mathcal{U})$

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Given $U \in \mathcal{U}$ the star of U against the cover $\mathcal{U}$ is the union of all the sets in $\mathcal{U}$ that intersect U.

Say that $\mathcal{U}$ **star refines** a cover $\mathcal{V}$ (denoted $\mathcal{U} \leq \mathcal{V}$) if, for any $U \in \mathcal{U}$, $star(U, \mathcal{U})$ is contained in some element of $\mathcal{V}$.

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For a set X a **uniform structure** is a collection of covers $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ which is a cofinal filter under reverse star refinement, and is closed under coarsening of the covers.

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A paracompact space X has a uniform structure. If $\mathcal{U}$ is an open cover of X then one can find an open cover $\mathcal{V}$ which is a star refinement of $\mathcal{U}$.

$\rightarrow$

Uniform structure and inverse limits I

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If $\Lambda = \{\mathcal{U}_n\}$ is a normal sequence for X, then one can define a pseudo-metric d on X:

- For x, y in X, let $n(x,y)$ be the largest n such that there exists $U \in \mathcal{U}_n$ containing both x and y.
- Let $\rho(x,y) = 2^{-n(x,y)}$, with the understanding that $2^{-\infty} = 0$.
- Let $d(x,y) = \inf \sum_{i=1}^{n} \rho(x_i, x_{i+1})$ where $x_1 = x$, $x_n = y$ and $x_i \in X$.

$\rightarrow$

Uniform structure and inverse limits II

On the previous slide we explained how to get a pseudo-metric d from a normal sequence of covers on X.

Let X_λ be the quotient of X obtained from the equivalence relation $x \sim y$ if and only $d(x,y) = 0$. The resulting space X_λ is a metric space, and $X \to X_\lambda$ is continuous.

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WARNING: We will refer to X_λ as a quotient of X; however, we warn that the topology of X_λ is not the canonical quotient topology induced by the original topology on X and the equivalence relation, but is instead determined by the choice of covers $\{\mathcal{U}_n\}$ (i.e. a possibly weaker topology).

$\rightarrow$

Uniform structure and inverse limits III

On the previous slide we explained how to get a metrizable quotient X_λ from a topological space X equipped with a normal sequence of covers.

Let $\mathcal{U}$ and $\mathcal{V}$ be two normal sequences for X. Say that $\mathcal{V}$ cofinally refines $\mathcal{U}$ if for every $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that $\mathcal{V}_{k(n)} \leq \mathcal{U}_n$.

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Let $\hat{X}_\alpha = \langle X, \mathcal{U} \rangle$ and $\hat{X}_\beta = \langle X, \mathcal{V} \rangle$ (meaning X equipped with the pseudo-metric resulting from $\mathcal{U}$ and $\mathcal{V}$ respectively). If $\mathcal{V}$ cofinally refines $\mathcal{U}$ then:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \hat{X}_\alpha \\ & \hat{X}_\beta \arrow[ur, "\varphi = id"'] & \end{tikzcd}$$

where the map $\varphi : \hat{X}_\beta \to \hat{X}_\alpha$ is uniformly continuous. If we then define X_α and X_β to be the corresponding quotient spaces, it should be clear that φ then induces a map $X_\beta \to X_\alpha$.

Uniform structure and inverse limits IV

Suppose additionally that $\boxed{X \text{ is Lindelöf}}$ – that is, that every open cover has a countable subcover. This assumption ensures that we can take each cover in the normal sequence to be countable, and thus that the space X_λ is second countable.

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We also note that for any open cover $\mathcal{U}$ of X we can get a normal sequence $\{\mathcal{U}_n\}$ for X such that $\mathcal{U}_1 = \mathcal{U}$.

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Considering all sequences of covers ordered by cofinal refinement we get:

Theorem

If X is a locally compact and Lindelöf space then it is the inverse limit of a system $\{X_\varphi, p^\varphi_\psi : X_\varphi \to X_\psi\}_{\varphi\in\Lambda}$ where each X_φ is second countable and locally compact, and all the connecting maps are proper.

Groupoids: Definition

One can think of a groupoid $\mathcal{G}$ as a set (of arrows), with two operations:

- a partial multiplication $(g,h) \mapsto gh$ defined on a subset $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$
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The set $\mathcal{G}^{(2)}$ is called the set of composable arrows. One defines $s,t : \mathcal{G} \to \mathcal{G}$ by $s(g) = gg^{-1}$ and $t(g) = g^{-1}g$ (the source and target of each arrow). The following rules must be satisfied:

- $(g,h) \in \mathcal{G}^{(2)}$ if and only if $t(g) = s(h)$, and composition of arrows is associative.
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- $s(g)$ and $t(g)$ act as identity elements for arrows with which they are composable.

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The image of the source map (or, equivalently, the target map) is a subset of $\mathcal{G}$ denoted $\mathcal{G}^{(0)}$, referred to as the **unit space** of the groupoid.

$\rightarrow$

Toplogical groupoids

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Of course, this implies the source and target maps are also continuous.

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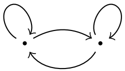
Examples: Topological Groupoids

-  with discrete topology

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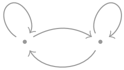
-  with discrete topology

- transformation groupoid - starting from G a topological group, X a topological space, and $\alpha : G \curvearrowleft X$ a continuous action of G on X by homeomorphisms. The groupoid is basically $X \times G$:

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[rr, bend right, "hg"'] & \alpha_g(x) \arrow[r, bend left, "h"] & \alpha_{hg}(x) \end{tikzcd}$$

$\rightarrow$

Groupoid Notation and Terminology

for $x \in \mathcal{G}^{(0)}$ write:

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Terminology: we say $\mathcal{G}$ is

- **open** if the source and target maps are open maps.
- **étale** if the source and target maps are local homeomorphisms.
- **transitive** if for any $x, y \in \mathcal{G}^{(0)}$ there exists $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$.

$\rightarrow$

Approximation for Groupoids I

Lemma

Let G be an open Lindelöf groupoid, and $\{K_n\}$ an exhaustion of G by compact sets. There exists a sequence of countable and locally finite open coverings $\{\mathcal{U}_n\}_{n\geq 0}$ of G such that for all $n \geq 0$:

1. *each set in $\mathcal{U}_n$ is pre-compact.*
2. $\mathcal{U}^0_{n+1} \leq s(\{K_n \cap U : U \in \mathcal{U}^1_n\}), t(\{K_n \cap U : U \in \mathcal{U}^1_n\}) \leq \mathcal{U}^0_n$.
3. $m(\mathcal{U}^1_{n+1}\big|_{K_n}, \mathcal{U}^1_{n+1}\big|_{K_n}) \leq \mathcal{U}^1_n$.
4. $(\mathcal{U}^1_{n+1})^{-1} \leq \mathcal{U}^1_n$.

$\rightarrow$

Approximation for Groupoids II

If we have a normal sequence of covers $\{\mathcal{U}_n\}$ for $\mathcal{G}$ satisfying the conditions of the previous slide, then we can form the quotient $\mathcal{G}_\alpha$ (in the same way as described for a general topological space).

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The extra conditions we impose on the normal sequence mean that we can define:

- $s([g]) = [s(g)]$ (where $g \in \mathcal{G}$ is a representative of $[g] \in \mathcal{G}_\alpha$)
- $[g], [h] \in \mathcal{G}_\alpha^{(2)}$ are composable if there exists $g' \in [g]$ and $h' \in [h]$ with $s(h') = t(g')$, in which case $m([g],[h]) = [g'h']$
- $[g]^{-1} = [g^{-1}]$

and these operations are well-defined and continuous in $\mathcal{G}_\alpha$.

$\rightarrow$

Haar system of measures

Let $\mathcal{G}$ be a topological groupoid. A **Haar system of measures on** $\mathcal{G}$ is a collection $\{\mu_x : x \in \mathcal{G}^{(0)}\}$ of positive Radon measures on $\mathcal{G}$ such that:

1. μ_x is supported on $\mathcal{G}^x$
2. for fixed $f \in \mathcal{C}_c(\mathcal{G})$, $x \mapsto \int_{\mathcal{G}^{(0)}} f(y)\, d\mu^x(y)$ is continuous on $\mathcal{G}^{(0)}$
3. for all $g \in \mathcal{G}^{(1)}$ and $f \in \mathcal{C}_c(\mathcal{G})$,

$$\int_{\mathcal{G}^{t(g)}} f(y)\, d\mu^{t(g)}(y) = \int_{\mathcal{G}^{s(g)}} f(gy)\, d\mu^{s(g)}(y).$$

The last condition is the groupoid equivalent of 'left invariance for Haar measure' in the case of groups. The second condition is a continuity condition for the choice of measures, and is needed in order for the convolution product to work (we will discuss this later).

$\longrightarrow$

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Having a Haar system on $\mathcal{G}$ enables one to construct the groupoid C^*-algebra (as explained later).

$\rightarrow$

Approximation of Haar system

We modify the construction of the normal sequence $\mathcal{U}_n$ by the addition of a partition of unity and the following condition:

> *Fix* $\{f^n_\omega : \omega \in \Lambda_n\}$ *a finite partition of unity of* K_n *whose carriers refine* $\mathcal{U}_n$*. Let* $(\lambda_\omega)_\omega \subset \mathbb{C}$ *be any sequence with* $|\lambda_\omega| < n$*. For each element* $U \in \mathcal{U}_{n+1}$ *and for each* $x, y \in s(U)$ *we have*

$$\left| \int_{\mathcal{G}} \left(\sum_{\omega} \lambda_\omega f^n_\omega \right) d\mu^x - \int_{\mathcal{G}} \left(\sum_{\omega} \lambda_\omega f^n_\omega \right) d\mu^y \right| < \frac{1}{n}.$$

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Let $q : \mathcal{G} \to \mathcal{G}_\alpha$. It follows that for each $f \in \mathcal{C}_c(\mathcal{G}_\alpha)$

$$x \sim y \text{ for } x, y \in \mathcal{G}^{(0)} \Rightarrow \int_{\mathcal{G}^x} (f \circ q)\, d\mu^x = \int_{\mathcal{G}^y} (f \circ q)\, d\mu^y,$$

allowing us to define a Haar system of measures on $\mathcal{G}_\alpha$ based on the Haar system of measures on $\mathcal{G}$.

$\rightarrow$

Groupoid C^*-algebra I

Assume $\{\mu^x\}$ is a Haar system on $\mathcal{G}$.
Equip $\mathcal{C}_c(\mathcal{G})$ with a convolution product and an involution operation. One can then complete the resulting algebra to a C^*-algebra (in fact, there is a reduced C^*-algebra and a full C^*-algebra).

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For $\varphi, \psi \in \mathcal{C}_c(\mathcal{G})$ define:
convolution: $(\varphi * \psi)(g) = \int \varphi(gh)\psi(h^{-1})\, d\mu^{s(g)}(h)$
involution: $\varphi^*(g) = \overline{\varphi(g^{-1})}$

We omit the description of the reduced and full C^*-algebra completion.

$\rightarrow$

Groupoid C^*-algebra II

With the construction described so far, $\mathcal{C}_c(\mathcal{G}_\alpha) \hookrightarrow \mathcal{C}_c(\mathcal{G})$ (as a *-algebra embedding).

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This is an easy computational check:
Denote by q the map $\mathcal{G} \to \mathcal{G}_\alpha$ which takes g to $[g]$. The embedding is given by $\tilde{q}(\varphi) = (\varphi \circ q)$ for $\varphi \in \mathcal{C}_c(\mathcal{G}_\alpha)$:

$$\begin{aligned}\tilde{q}(\varphi * \psi)(g) = (\varphi * \psi)(q(g)) &= \int_{\mathcal{G}_\alpha} \varphi(q(g)q(h))\psi(q(h)^{-1})\, d\mu^{s(q(g))}(q(h)) \\ &= (\tilde{q}(\varphi) * \tilde{q}(\psi))(g)\end{aligned}$$

$$\tilde{q}(\varphi^*)(g) = (\varphi^* \circ q)(g) = \overline{\varphi(q(g)^{-1})} = \overline{\tilde{q}(\varphi)(g^{-1})} = (\tilde{q}(\varphi))^*(g),$$

where $\varphi, \psi \in \mathcal{C}_c(\mathcal{G}_\alpha)$and $g \in \mathcal{G}$. We used the fact that $q : \mathcal{G} \to \mathcal{G}_\alpha$ respects the groupoid operations and is onto.

$\rightarrow$

2-Cocycles and twisted convolution algebra

A 2-cocycle for $\mathcal{G}$ is a map $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ such that

- $\sigma(g,h)\sigma(gh,k) = \sigma(g,hk)\sigma(gh,k)$ for all $(g,h),(h,k) \in \mathcal{G}^{(2)}$, and
- $\sigma(g,s(g)) = 1 = \sigma(t(g),g)$ for all $g \in \mathcal{G}$.

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A cocycle on $\mathcal{G}$ can be pushed to a cocycle on $\mathcal{G}_\alpha$ by again modifying the normal sequence construction to add the following condition:

> *Choose a normal sequence $\{\mathcal{V}_n\}$ for $\mathbb{T}$, where $\mathcal{V}_n$ is a finite cover by $\frac{1}{2^n}$-balls, and also ask that the sequence $\{\mathcal{U}_n\}$ satisfies:*

$$\sigma(\,\mathcal{U}_n|_{K_n},\,\mathcal{U}_n|_{K_n}) \leq \mathcal{V}_n$$

This ensures that we can define $\sigma([g],[h]) = \sigma(g,h)$ for $g,h \in G$ representatives of $[g],[h] \in G_\alpha$. Similarly to the previous slide, $\mathcal{C}_c(\mathcal{G}_\alpha,\sigma) \hookrightarrow \mathcal{C}_c(\mathcal{G},\sigma)$ (as a *-algebra embedding).

$\rightarrow$

Disintegration Theorem

The version of Renault's theorem mentioned below omits the mention of a 2-cocycle for $\mathcal{G}$ in order to simplify slightly the presentation.

Theorem (Renault's Disintegration Theorem)

*Let $\mathcal{G}$ be a second countable locally compact groupoid endowed with a Haar system of measures. Every nondegenerate representation of the *-algebra $\mathcal{C}_c(\mathcal{G})$ on a separable Hilbert space is the integrated form of a representation of $\mathcal{G}$ on a bundle of Hilbert spaces.*

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Our goal is to use the approximation by second countable groupoids described in the earlier part of the talk to bootstrap this result to σ-compact groupoids.

$\rightarrow$

Hilbert bundles

A **Borel bundle of Hilbert spaces over a space** X is a Borel space $Z = X * \mathcal{H} = \{(x, v) : x \in X \text{ and } v \in \mathcal{H}_x\}$ such that

1. the projection $p : Z \to X$ is measurable, and each fiber $\mathcal{H}_x$ is a Hilbert space

along with measurable sections $\{s_\alpha\}_{\alpha \in A}$ to p such that for each $x \in X$ the span of the set $\{s_\alpha(x) : \alpha \in A\}$ is dense in $p^{-1}(x)$ and satisfying the following properties

1. for each $\alpha \in A$ the map $(x, v) \to \langle s_\alpha(x), v\rangle$ is measurable on Z.
2. for each $\alpha, \beta \in A$ the map $x \to \langle s_\alpha(x), s_\beta(x)\rangle_{\mathcal{H}_x}$ is measurable on X.
3. the functions $(x, v) \to \langle s_\alpha(x), v\rangle$ separate the points of Z.

If $A = \mathbb{N}$ then we say that the bundle is **separable**.

$\rightarrow$

Representations of Groupoids

We represent groupoids on a bundle of Hilbert spaces.

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Isomorphism groupoid: $Iso(\mathcal{G}^{(0)} * \mathcal{H}) := \{(x, U, y) : U : \mathcal{H}_x \to \mathcal{H}_y$ is unitary$\}$.

Representations of Groupoids

We represent groupoids on a bundle of Hilbert spaces.

Isomorphism groupoid: $Iso(\mathcal{G}^{(0)} * \mathcal{H}) := \{(x,U,y) : U : \mathcal{H}_x \to \mathcal{H}_y \text{ is unitary}\}$.

If $\mathcal{G}$ is a groupoid equipped with a Haar system of measures, a **unitary representation** of $\mathcal{G}$ is a triple $(\mathcal{G}^{(0)} * \mathcal{H}, L, \nu)$ where

- $\mathcal{G}^{(0)} * \mathcal{H}$ is a Borel Hilbert bundle over $\mathcal{G}^{(0)}$
- $L : \mathcal{G} \to Iso(\mathcal{G}^{(0)} * \mathcal{H})$ is such that $L(g) = (t(g), L_g, s(g))$
- ν is a quasi-invariant measure on $\mathcal{G}^{(0)}$;

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additionally define Borel sections and square-integrable sections
$B(\mathcal{G}^{(0)} * \mathcal{H}) := \{f : \mathcal{G}^{(0)} \to \mathcal{G}^{(0)} * \mathcal{H} : x \mapsto \langle f(x), f_\alpha(x)\rangle \text{ is Borel for all } \alpha\}$
$L^2(\mathcal{G}^{(0)} * \mathcal{H}, \nu) = \{f \in B(\mathcal{G}^{(0)} * \mathcal{H}) : x \mapsto \|f(x)\|^2 \text{ is integrable on } \mathcal{G}^{(0)}\}$,

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and impose the condition that $g \mapsto \langle L_g h(s(g)), k(t(g))\rangle$ should be ν-measurable for all $h, k \in L^2(\mathcal{G}^{(0)} * \mathcal{H}, \nu)$.

$\rightarrow$

Integrated form of a representation

Suppose $\mathcal{G}$ is a groupoid equipped with a Haar system of measures $\{\mu^x\}$. Suppose moreover that $(\mathcal{G}^{(0)} * \mathcal{H}, L, \nu)$ is a unitary representation of $\mathcal{G}$.

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There is a representation of $\mathcal{C}_c(\mathcal{G})$, called the integrated form of $(\mathcal{G}^{(0)} * \mathcal{H}, L, \nu)$, denoted by L through a standard abuse of notation, defined such that

$$\langle L(\varphi)h, k\rangle = \int_{\mathcal{G}^{(0)}} \int_G \varphi(g) \langle L_g(h(s(g))), k(t(g))\rangle \, \Delta(g)^{-1/2} \, d\mu^x(g) \, d\nu(x),$$

where $\varphi \in \mathcal{C}_c(\mathcal{G})$, $h, k \in L^2(\mathcal{G}^{(0)} * \mathcal{H}, \nu)$ and Δ is the modular function of ν.

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Renault's disintegration theorem states that if $\mathcal{G}$ is second countable then all representations of $\mathcal{C}_c(\mathcal{G})$ are of this type.

$\rightarrow$

Disintegration Theorem for Lindelöf groupoids

We want to extend Renault's result to Lindelöf groupoids:

Theorem

*Let $\mathcal{G}$ be a Lindelöf locally compact groupoid endowed with a Haar system of measures. Every nondegenerate representation of the *-algebra $\mathcal{C}_c(\mathcal{G})$ on a Hilbert space is the integrated form of a representation of $\mathcal{G}$ on a bundle of Hilbert spaces.*

$\rightarrow$

Comments, and avenues for future investigation

Some obvious comments about the quotient construction:

- If $\mathcal{G}$ is transitive, so is $\mathcal{G}_\alpha$.
- If $\mathcal{G}$ is étale, it is easy to ensure the quotient groupoids $\mathcal{G}_\alpha$ are also étale.

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Some obvious comments about the quotient construction:

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Some questions:

- Is it true that if $\mathcal{G}$ is étale then finite dynamic asymptotic dimension is preserved by the construction?
- If $\mathcal{G}$ is equipped with a Fell bundle, is there a good way to associate a Fell bundle to $\mathcal{G}_\alpha$?
- are there results for second countable groupoids that, using the ideas / constructions presented, can be extended to Lindelöf groupoids?

$\rightarrow$

...The End

Thank you for your attention.