Pick interpolation and the displacement equation

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Multivariable Operator Theory at the Technion
On the occasion of Baruch Solel’s 65th birthday
Classical Nevanlinna-Pick theorem

Let $H^\infty(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is bounded and analytic} \}$.

**Theorem (Pick 1915)**

Given $N$ distinct points $z_1, \ldots, z_N \in \mathbb{D}$ and $N$ points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, there exists $f \in H^\infty(\mathbb{D})$ such that $\|f\|_\infty \leq 1$ and

$$f(z_i) = \lambda_i, \quad i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\begin{bmatrix} 1 - \bar{\lambda}_i \lambda_j & N \\ \frac{1 - \bar{z}_i z_j}{1 - \bar{z}_i z_j} & i,j=1 \end{bmatrix}$$

is positive semidefinite.
Early generalizations

- (Nagy-Koranyi 1956) $\lambda_i \in M_n(\mathbb{C})$.
- (Sarason 1967) Commutant lifting in $H^\infty(\mathbb{D})$ implies classical Nevanlinna-Pick theorem and Nagy-Koranyi theorem.
- (Ball-Gohberg 1985) Commutant lifting in the set of block upper triangular matrices implies Nevanlinna-Pick theorem for $z_i \in M_n(\mathbb{C})$ and $\lambda_i \in M_m(\mathbb{C})$. 
Generalizations of interest

Two main strategies for proving generalized noncommutative Nevanlinna-Pick theorems since 1967:

- displacement equation
- commutant lifting
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- displacement equation
- commutant lifting

Goal:

- Understand the relationship between these two approaches
Generalizations of interest

Two main strategies for proving generalized noncommutative Nevanlinna-Pick theorems since 1967:

- displacement equation (Constantinescu-Johnson 2003)

Goal:

- Understand the relationship between these two approaches
**W*-algebra**

A **W*-algebra** $M$ is a $C^*$-algebra that is a dual space.
**$W^*$-algebra**

A $W^*$-algebra $M$ is a $C^*$-algebra that is a dual space.

**$W^*$-correspondence**

A $W^*$-correspondence $E$ over a $W^*$-algebra $M$ is

- a Hilbert $C^*$-module over $M$;
- self-dual;
- equipped with a faithful, normal $\ast$-homomorphism $\varphi : M \to \mathcal{L}(E)$ that gives the left action of $M$ on $E$. 
Examples of $W^*$-correspondences

- $M = E = \mathbb{C}$
  - $a \cdot c \cdot b = acb$
  - $\langle c, d \rangle = \overline{cd}$

- $M = \mathbb{C}$, $E = \mathbb{C}^n$
  - $a \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \cdot b = \begin{bmatrix} ac_1 b \\ \vdots \\ ac_n b \end{bmatrix}$
  - $\langle \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \rangle = \sum \overline{c_i} d_i$
Examples cont.

- \( G = (G^0, G^1, r, s), \ M = C(G^0), \ E = C(G^1) \)
- \( (a \cdot \xi \cdot b)(e) = a(r(e))\xi(e)b(s(e)) \)
- \( \langle \xi, \eta \rangle(v) = \sum_{s(e) = v}^{\xi(e)\eta(e)} \)
Given

- $M$, a $W^*$-algebra
- $E$, a $W^*$-correspondence over $M$

define

- the **Fock space** $\mathcal{F}(E)$ to be the ultraweak direct sum $\bigoplus_{k=0}^{\infty} E^\otimes k$, where $E^\otimes 0 = M$, viewed as a bimodule over itself
- the von Neumann algebra of bounded operators $\mathcal{L}(\mathcal{F}(E))$ on the Fock space of $E$
Define the **left action operator** $\varphi_\infty : M \to \mathcal{L}(\mathcal{F}(E))$ by

$$
\varphi_\infty(a) = \begin{bmatrix}
a \\
\varphi(a) \\
\varphi_2(a) \\
\vdots
\end{bmatrix}
$$

where $\varphi_k(a) : E \otimes^k \to E \otimes^k$ is given by

$$
\varphi_k(a)(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_k) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \ldots \otimes \xi_k.
$$
For $\xi \in E$, define the **left creation operator** $T_{\xi} : \mathcal{F}(E) \to \mathcal{F}(E)$ by $T_{\xi}(\eta) = \xi \otimes \eta$, i.e.,

\[
T_{\xi} = \begin{bmatrix} 0 & T_{\xi}^{(1)} & 0 \\ T_{\xi}^{(1)} & 0 & T_{\xi}^{(2)} \\ & \ddots & \ddots \end{bmatrix}
\]

where $T_{\xi}^{(k)} : E \otimes^{k-1} \to E \otimes^{k}$ is given by

\[
T_{\xi}^{(k)}(\eta_1 \otimes \ldots \otimes \eta_{k-1}) = \xi \otimes \eta_1 \otimes \ldots \otimes \eta_{k-1}.
\]
Subalgebras of $\mathcal{L}(\mathcal{H}(E))$

**Tensor algebra of $E$**

The **tensor algebra** of $E$, denoted $\mathcal{T}_+(E)$, is the norm-closed subalgebra of $\mathcal{L}(\mathcal{H}(E))$ generated by $\{\varphi_\infty(a) \mid a \in M\}$ and $\{T_\xi \mid \xi \in E\}$. 
Subalgebras of $\mathcal{L}(\mathcal{F}(E))$

Tensor algebra of $E$

The **tensor algebra** of $E$, denoted $\mathcal{T}_+(E)$, is the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\{\varphi_\infty(a) \mid a \in M\}$ and $\{T_\xi \mid \xi \in E\}$.

Hardy algebra of $E$

The **Hardy algebra** of $E$, denoted $\mathcal{H}_\infty(E)$, is the ultraweak closure of $\mathcal{T}_+(E)$ in $\mathcal{L}(\mathcal{F}(E))$. 
The $\sigma$-dual $E^\sigma$

Given

- $M$, a $W^*$-algebra
- $E$, a $W^*$-correspondence
- $\sigma : M \to B(H)$, a faithful, normal representation of $M$ on a Hilbert space $H$,

define

- $E^\sigma := \{ \eta \in B(H, E \otimes_\sigma H) \mid \eta \sigma(a) = (\varphi(a) \otimes I_H)\eta \ \forall a \in M \}$. 
$E^\sigma$ is a $W^*$-correspondence over $\sigma(M)'$:

$$E^\sigma := \{ \eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I_H)\eta \forall a \in M \}$$

$E^\sigma$ is a $W^*$-correspondence over $\sigma(M)'$:

- $a \cdot \eta \cdot b := (I_E \otimes a)\eta b$
- $\langle \eta, \xi \rangle := \eta^*\xi$

Construct $H^\infty(E^\sigma)$.
Cauchy kernel

\[ E^\sigma := \{ \eta \in B(H, E \otimes_\sigma H) \mid \eta \sigma(a) = (\varphi(a) \otimes I_H) \eta \ \forall a \in M \} \]

For \( \eta \in E^\sigma \) with \( \|\eta\| < 1 \) and \( k \in \mathbb{N} \), define

- **the \( k \)th tensorial power** \( \eta^{(k)} \in B(H, E^{\otimes k} \otimes_\sigma H) \) by
  \[
  \eta^{(k)} = (I_{E^{\otimes k-1}} \otimes \eta)(I_{E^{\otimes k-2}} \otimes \eta) \cdots (I_E \otimes \eta) \eta
  \]

- **the Cauchy kernel** \( C(\eta) \in B(H, \mathcal{F}(E) \otimes_\sigma H) \) by
  \[
  C(\eta) = \begin{bmatrix} I_H & \eta & \eta^{(2)} & \eta^{(3)} & \ldots \end{bmatrix}^T
  \]
Spectral radius in the $\mathcal{W}^*$-correspondence setting

**Spectral radius**

For $\eta \in E^\sigma$, define the spectral radius of $\eta$ by

$$r(\eta) := \inf_k \|\eta^{(k)}\|^{1/k}.$$
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**Proposition**

For $\eta \in E^\sigma$, $C(\eta) \in B(H, \mathcal{F}(E) \otimes_\sigma H)$ if and only if $r(\eta) < 1$. 
Spectral radius in the $W^*$-correspondence setting

**Spectral radius**

For $\eta \in E^\sigma$, define the **spectral radius** of $\eta$ by

$$r(\eta) := \inf_k \|\eta^{(k)}\|^{1/k}.$$ 

**Proposition**

For $\eta \in E^\sigma$, $C(\eta) \in B(H, \mathcal{F}(E) \otimes_\sigma H)$ if and only if $r(\eta) < 1$. For $\eta \in E^\sigma$, $\|\eta\| < 1$ implies $r(\eta) < 1$, but the converse is not true.
Define $U : \mathcal{F}(E^\sigma) \otimes_l H \to \mathcal{F}(E) \otimes_\sigma H$ by

$$U(\eta_1 \otimes \cdots \otimes \eta_k \otimes h) = (I_{E \otimes k-1} \otimes \eta_1) \cdots (I_E \otimes \eta_{k-1}) \eta_k h.$$ 

Define $\rho : H^\infty(E^\sigma) \to B(\mathcal{F}(E) \otimes_\sigma H)$ by

$$\rho(X) = U(X \otimes I_H)U^*.$$
Point evaluation

Define $U : \mathcal{F}(E^\sigma) \otimes H \to \mathcal{F}(E) \otimes H$ by

$$U(\eta_1 \otimes \cdots \otimes \eta_k \otimes h) = (l_{E^{\otimes k-1}} \otimes \eta_1) \cdots (l_E \otimes \eta_{k-1})\eta_k h.$$ 

Define $\rho : H^\infty(E^\sigma) \to B(\mathcal{F}(E) \otimes H)$ by

$$\rho(X) = U(X \otimes I_H)U^*.$$ 

For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $r(\eta) < 1$, define the point evaluation $\hat{X}(\eta)$ by

$$\hat{X}(\eta) = C(0)^* \rho(X)^* C(\eta).$$
Remarks about the point evaluation

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- $\hat{X}(\eta) \in \sigma(M)'$
Remarks about the point evaluation

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- $\hat{X}(\eta) \in \sigma(M)'$
- Not multiplicative, i.e., $\hat{XY}(\eta) \neq \hat{X}(\eta) \hat{Y}(\eta)$
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- $\hat{X}(\eta) \in \sigma(M)'$
- Not multiplicative, i.e., $\hat{XY}(\eta) \neq \hat{X}(\eta) \hat{Y}(\eta)$
- Induces an algebra antihomomorphism from $H^\infty(E^\sigma)$ into the completely bounded maps on $\sigma(M)'$
Muhly-Solel point evaluation (Muhly-Solel 2004)

For $Y \in H^{\infty}(E)$ and $\eta \in E^{\sigma}$ with $\|\eta\| < 1$, define the point evaluation $\hat{Y}(\eta^*)$ by

$$\hat{Y}(\eta^*) = (C(0)^*(Y^* \otimes I_H)C(\eta))^*.$$
Muhly-Solel point evaluation

Muhly-Solel point evaluation (Muhly-Solel 2004)

For \( Y \in H^\infty(E) \) and \( \eta \in E^\sigma \) with \( \|\eta\| < 1 \), define the point evaluation \( \hat{Y}(\eta^*) \) by

\[
\hat{Y}(\eta^*) = (C(0)^*(Y^* \otimes I_H)C(\eta))^*.
\]

- \( \hat{Y}(\eta^*) \in B(H) \)
Definitions

Generalized Nevanlinna-Pick theorem
Comparison with Popescu’s theorem
Comparison with Muhly-Solel’s theorem

Muhly-Solel point evaluation

Muhly-Solel point evaluation (Muhly-Solel 2004)

For \( Y \in H^\infty(E) \) and \( \eta \in E^\sigma \) with \( \|\eta\| < 1 \), define the point evaluation \( \hat{Y}(\eta^*) \) by

\[
\hat{Y}(\eta^*) = (C(0)^*(Y^* \otimes I_H)C(\eta))^*.
\]

- \( \hat{Y}(\eta^*) \in B(H) \)
- Multiplicative, i.e., \( \hat{X}Y(\eta^*) = \hat{X}(\eta^*)\hat{Y}(\eta^*) \)
Theorem (N.)

Let $z_1, \ldots, z_N$ be $N$ distinct elements of $E^\sigma$ with $r(z_i) < 1$ for all $i$, and let $\Lambda_1, \ldots, \Lambda_N \in \sigma(M)'$. There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that

$$\hat{X}(z_i) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the operator matrix

$$A_N = \left[ C(z_i)^* (I_{\mathcal{F}(E)} \otimes (I_H - \Lambda_i^* \Lambda_j)) C(z_j) \right]_{i,j=1}^N$$

is positive semidefinite.
Corollary: Classical Nevanlinna-Pick theorem

If

- $M = E = \mathbb{C}$
- $\sigma : M \rightarrow B(\mathbb{C})$ is given by $\sigma(a) = a$

then

- $E^\sigma = \mathbb{C}$
- $\sigma(M)' = \mathbb{C}$
- we recover the classical Nevanlinna-Pick theorem
Corollary: Constantinescu-Johnson’s theorem

If

- $M = \mathbb{C}$, $E = \mathbb{C}^n$
- $\sigma : M \rightarrow B(H)$ is given by $\sigma(a) = aI_H$

then

- $E^\sigma = C_n(B(H))$
- $\sigma(M)' = B(H)$

we recover Constantinescu-Johnson’s theorem.
A \textbf{displacement equation} is an equation of the form

\[(I_{B(H)} - \theta)(A) = B,\]

where \(A, B \in B(H)\) and \(\theta : B(H) \to B(H)\).
A displacement equation is an equation of the form

$$(l_{B(H)} - \theta)(A) = B,$$

where $A, B \in B(H)$ and $\theta : B(H) \to B(H)$. Given $\theta$ and $B$, and assuming $(l_{B(H)} - \theta)^{-1}$ exists, solve for $A$:

$$A = (l_{B(H)} - \theta)^{-1}(B) = \sum_{k=0}^{\infty} \theta^k(B).$$
A displacement equation is an equation of the form

\[(I_{B(H)} - \theta)(A) = B,\]

where \(A, B \in B(H)\) and \(\theta : B(H) \to B(H)\).

Given \(\theta\) and \(B\), and assuming \((I_{B(H)} - \theta)^{-1}\) exists, solve for \(A\):

\[A = (I_{B(H)} - \theta)^{-1}(B) = \sum_{k=0}^{\infty} \theta^k(B).\]

We are interested in the case when \(\theta\) is completely positive. In this case, \((I_{B(H)} - \theta)^{-1}\) is completely positive as well.
Proof of (N.)

Step 1

Let $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}$, $U = \begin{bmatrix} I_H \\ \vdots \\ I_H \end{bmatrix}$, and $V = \begin{bmatrix} \Lambda_1^* \\ \vdots \\ \Lambda_N^* \end{bmatrix}$. Form the displacement equation

$$(I_{B(H)} - \theta_\mathbf{z})(A) = UU^* - VV^*,$$

where $A \in B(H)$ and $\theta_\mathbf{z}(A) = \mathbf{z}^*(I_E \otimes A)\mathbf{z}$. 

Definitions

Generalized Nevanlinna-Pick theorem
Comparison with Popescu’s theorem
Comparison with Muhly-Solel’s theorem
Step 2

Observe

- The Pick matrix is the unique solution of the displacement equation, i.e.,

\[ A = A_N \]
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Observe

- The Pick matrix is the unique solution of the displacement equation, i.e.,
  \[ A = A_N \]

- We can rewrite the Pick matrix as \( A_N = U_\infty^* U_\infty - V_\infty^* V_\infty \), where
  \[ U_\infty = \begin{bmatrix} C(\delta_1) & \cdots & C(\delta_N) \end{bmatrix} \]
  and
  \[ V_\infty = \begin{bmatrix} (I_{\mathcal{F}(E)} \otimes \Lambda_1)C(\delta_1) & \cdots & (I_{\mathcal{F}(E)} \otimes \Lambda_N)C(\delta_N) \end{bmatrix}. \]
Lemma (Step 3)

\[ A_N = U_\infty^* U_\infty - V_\infty^* V_\infty \] is positive semidefinite if and only if there exists \( X \in H^\infty(E^\sigma) \) with \( \|X\| \leq 1 \) such that \( \rho(X)^* U_\infty = V_\infty \).
Lemma (Step 3)

\[ \mathcal{A}_N = U_\infty^* U_\infty - V_\infty^* V_\infty \] is positive semidefinite if and only if there exists \( X \in H^\infty(E^\sigma) \) with \( \|X\| \leq 1 \) such that \( \rho(X)^* U_\infty = V_\infty \).

Step 4

\( \rho(X)^* U_\infty = V_\infty \) if and only if \( \hat{X}(\hat{z}_i) = \Lambda_i \) for all \( i \).
Proof of lemma

Lemma

\( \mathcal{A}_\mathcal{N} = U_\infty^* U_\infty - V_\infty^* V_\infty \) is positive semidefinite if and only if there exists \( X \in H^\infty(E^\sigma) \) with \( \|X\| \leq 1 \) such that \( \rho(X)^* U_\infty = V_\infty \).

Proof: ( \( \Rightarrow \) )

- \( \mathcal{A}_\mathcal{N} \geq 0 \Rightarrow \exists L \in \sigma^{(N)}(M)' \) such that \( \mathcal{A}_\mathcal{N} = LL^* \)
- Displacement equation becomes \( \hat{A}^* \hat{A} = \hat{B}^* \hat{B} \)
Proof of lemma cont.

- Douglas’s lemma $\implies \exists$! partial isometry $\Omega$ such that
  - $\hat{A} = \Omega \hat{B}$
  - $\text{Inn}(\Omega) \subseteq \text{Range}(\hat{B})$
- Define matrix $T$ in terms of the entries of $\Omega$ so that $TU_\infty = V_\infty$.
- There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $T = \rho(X)^*$, and $\rho(X)^* U_\infty = V_\infty$. 
Proof of lemma cont.

\[(\Leftarrow)\text{ If there exists } X \in \mathcal{H}^\infty(E^\sigma) \text{ such that } \|X\| \leq 1 \text{ and } \\
\rho(X)^* U_\infty = V_\infty, \text{ then } \]

\[
\mathcal{A}_N = U_\infty^* U_\infty - V_\infty^* V_\infty \\
= U_\infty^* U_\infty - U_\infty^* \rho(X) \rho(X)^* U_\infty \\
= U_\infty^* (I - \rho(X) \rho(X)^*) U_\infty \\
\geq 0
\]

since \( \|X\| \leq 1 \) and \( \rho \) is an isometry.
Popescu’s setting

- $M = \mathbb{C}$
- $E = \mathbb{C}^n$
- $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$
- $E^\sigma = C_n(B(H))$
- $\sigma(M)' = B(H)$
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Spectral radius

For $\tilde{z} = [Z_1 \cdots Z_n] \in B(H)^n$, the spectral radius of $\tilde{z}$ is given by

$$r(\tilde{z}) := \inf_k \left\| \sum_{|\alpha|=k} Z_\alpha (Z_\alpha)^* \right\|^{1/2k}$$
Popescu’s setting

- $M = \mathbb{C}$
- $E = \mathbb{C}^n$
- $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$
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- $\sigma(M)' = B(H)$

**Spectral radius**

For $\z = [Z_1 \ldots Z_n] \in B(H)^n$, the **spectral radius** of $\z$ is given by

$$r(\z) := \inf_k \left\| \sum_{|\alpha| = k} Z_\alpha Z_\alpha^* \right\|^{1/2k}$$

$$= \inf_k \| \z^*(k) \|^{1/k} = r(\z^*).$$
Let $S_1, \ldots, S_n$ be the **left creation operators** on the Fock space of $\mathbb{C}^n$. For $\Phi \in H^\infty(\mathbb{C}^n) \otimes B(H)$, we can write

$$
\Phi = \sum_{\alpha \in F_n^+} S_\alpha \otimes A(\alpha), \quad A(\alpha) \in B(H).
$$

**Point evaluation**

Define the **point evaluation** of $\Phi = \sum_{\alpha \in F_n^+} S_\alpha \otimes A(\alpha)$ at $\tilde{z} = [Z_1 \cdots Z_n]$ with $r(\tilde{z}) < 1$ by

$$
\Phi(\tilde{z}) := \sum_{\alpha \in F_n^+} Z_{\tilde{\alpha}} A(\alpha),
$$

where $\tilde{\alpha}$ denotes the reverse of $\alpha$. 
Theorem (Popescu 2003)

For \( i = 1, \ldots, N \), let \( \tilde{z}_i = [Z_{i1} \cdots Z_{in}] \in B(H)^n \) with \( r(\tilde{z}_i) < 1 \), and let \( \Lambda_i \in B(H) \). There exists \( \Phi \in H^\infty(\mathbb{C}^n) \otimes B(H) \) such that \( \|\Phi\| \leq 1 \) and

\[
\Phi(\tilde{z}_i) = \Lambda_i, \quad i = 1, \ldots, N,
\]

if and only if the operator matrix

\[
A_P = \left[ \sum_{k=0}^{\infty} \sum_{|\alpha| = k} Z_{i\alpha}(I_H - \Lambda_i \Lambda_j^*)(Z_{j\alpha})^* \right]_{i,j=1}^N
\]

is positive semidefinite.
Proof via the displacement equation

Given the data in Popescu’s theorem, (N.) implies that there exists \( X \in H^\infty(E^\sigma) \) such that \( \|X\| \leq 1 \) and \( \hat{X}(\beta_i^*) = \Lambda_i^* \) for all \( i \) if and only if the Pick matrix \( A_N \) is positive semidefinite.
Proof via the displacement equation

1. Given the data in Popescu’s theorem, (N.) implies that there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(z_i^*) = \Lambda_i^*$ for all $i$ if and only if the Pick matrix $A_N$ is positive semidefinite.

2. The Pick matrices $A_P$ and $A_N$ are equal.
Given the data in Popescu’s theorem, (N.) implies that there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\delta_i^*) = \Lambda_i^*$ for all $i$ if and only if the Pick matrix $A_N$ is positive semidefinite.

The Pick matrices $A_P$ and $A_N$ are equal.

There exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\delta_i^*) = \Lambda_i^*$ if and only if there exists $\Phi \in H^\infty(\mathbb{C}^n) \otimes B(H)$ such that $\|\Phi\| \leq 1$ and $\Phi(\delta_i) = \Lambda_i$. 

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Pick interpolation and the displacement equation
1. Given the data in Popescu’s theorem, (N.) implies that there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\lambda_i^*) = \Lambda_i^*$ for all $i$ if and only if the Pick matrix $A_N$ is positive semidefinite.

2. The Pick matrices $A_P$ and $A_N$ are equal.

3. There exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\lambda_i^*) = \Lambda_i^*$ if and only if there exists $\Phi \in H^\infty(\mathbb{C}^n) \otimes B(H)$ such that $\|\Phi\| \leq 1$ and $\Phi(\lambda_i) = \Lambda_i$. Hint: $\Phi = (J \otimes I_H)\rho(X)(J \otimes I_H)$, where $J : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ is given by $J(e_{i_1} \otimes \cdots \otimes e_{i_k}) = e_{i_k} \otimes \cdots \otimes e_{i_1}$. 
Theorem (Muhly-Solel 2004)

Let \( z_1, \ldots, z_N \) be \( N \) distinct points of \( E^\sigma \) with \( \|z_i\| < 1 \) for all \( i \), and let \( \Lambda_1, \ldots, \Lambda_N \in B(H) \). There exists \( Y \in H^\infty(E) \) with \( \|Y\| \leq 1 \) such that

\[
\hat{Y}(z_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N,
\]

if and only if the map from \( M_N(\sigma(M)'') \) to \( M_N(B(H)) \) defined by

\[
B \mapsto (I - \Psi_\Lambda) \circ (I - \theta_\zeta)^{-1}(B)
\]

is completely positive, where \( \Lambda = \text{diag}[\Lambda_i] \), \( \zeta = \text{diag}[z_i] \), \( \Psi_\Lambda(C) = \Lambda^* C \Lambda \), and \( \theta_\zeta(C) = \zeta^*(I_E \otimes C)\zeta \) for all \( C \in M_N(\sigma(M)'') \).
Theorem (Muhly-Solel 2004)

Let \( z_1, \ldots, z_N \) be \( N \) distinct points of \( E^\sigma \) with \( \|z_i\| < 1 \) for all \( i \), and let \( \Lambda_1, \ldots, \Lambda_N \in B(H) \). There exists \( Y \in H^\infty(E) \) with \( \|Y\| \leq 1 \) such that

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\hat{Y}(z_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N,
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if and only if the map from \( M_N(\sigma(M)'') \) to \( M_N(B(H)) \) defined by

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B \mapsto (I - \Psi_\Lambda) \circ (I - \theta_\hat{z})^{-1}(B)
\]

is completely positive, where \( \Lambda = \text{diag}[\Lambda_i] \), \( \hat{z} = \text{diag}[\hat{z}_i] \), \( \Psi_\Lambda(C) = \Lambda^* C \Lambda \), and \( \theta_\hat{z}(C) = \hat{z}^* (I_E \otimes C) \hat{z} \) for all \( C \in M_N(\sigma(M)'') \).
An implication

Suppose the Muhly-Solel Pick matrix map \((I - \Psi \Lambda) \circ (I - \theta_3)^{-1}\) is completely positive.

Moral: Interpolation in the sense of Muhly-Solel’s theorem implies interpolation in the sense of \((N.)\). However, a simple example shows that the converse is not true.
An implication

Suppose the Muhly-Solel Pick matrix map \((I - \Psi) \circ (I - \theta_3)^{-1}\) is completely positive.
\[ \implies (I - \theta_3)^{-1} \circ (I - \Psi) \] is completely positive.
Suppose the Muhly-Solel Pick matrix map \((I - \Psi) \circ (I - \theta_3)^{-1}\) is completely positive.

\[
\begin{align*}
&\implies (I - \theta_3)^{-1} \circ (I - \Psi) \text{ is completely positive.} \\
&\implies \mathcal{A}_N = (I - \theta_3)^{-1}(UU^* - VV^*)
\end{align*}
\]
An implication

Suppose the Muhly-Solel Pick matrix map \((I - \Psi) \circ (I - \theta_3)^{-1}\) is completely positive.

\[\implies (I - \theta_3)^{-1} \circ (I - \Psi)\] is completely positive.

\[\implies \mathcal{A}_\mathcal{N} = (I - \theta_3)^{-1}(UU^* - VV^*)\]

\[= (I - \theta_3)^{-1} \circ (I - \Psi) \begin{pmatrix} I_H & \cdots & I_H \\ \vdots & \ddots & \vdots \\ I_H & \cdots & I_H \end{pmatrix} \geq 0.\]
Suppose the Muhly-Solel Pick matrix map \((I - \Psi \Lambda) \circ (I - \theta_3)^{-1}\) is completely positive.

\[
\Rightarrow (I - \theta_3)^{-1} \circ (I - \Psi \Lambda) \text{ is completely positive.}
\]

\[
\Rightarrow A_N = (I - \theta_3)^{-1}(UU^* - VV^*)
\]

\[
= (I - \theta_3)^{-1} \circ (I - \Psi \Lambda) \begin{pmatrix} I_H & \cdots & I_H \\ \vdots & \ddots & \vdots \\ I_H & \cdots & I_H \end{pmatrix} \geq 0.
\]

**Moral:** Interpolation in the sense of Muhly-Solel’s theorem implies interpolation in the sense of \((N.)\). However, a simple example shows that the converse is not true.
Theorem (N. 2017)

Let \( \delta_1, \ldots, \delta_N \) be \( N \) distinct elements of \( \mathcal{Z}(E^\sigma) \) with \( \| \delta_i \| < 1 \) for all \( i \), and let \( \Lambda_1, \ldots, \Lambda_N \in \mathcal{Z}(\sigma(M)' \).
Let $\hat{Y}(\hat{z}_i^*) = \Lambda_i^*$, $i = 1, \ldots, N$ in the sense of (Muhly-Solel 2004).

2 There exists $X \in H^\infty(\mathcal{Z}(E^\sigma))$ with $\|X\| \leq 1$ such that $\hat{X}(\hat{z}_i) = \Lambda_i$, $i = 1, \ldots, N$ in the sense of (N.).
Definitions

Center of a $W^*$-correspondence

$\mathcal{Z}(E) := \{\xi \in E \mid a \cdot \xi = \xi \cdot a \quad \forall a \in M\}$

If $(M, E)$ is a $W^*$-correspondence, then $(\mathcal{Z}(M), \mathcal{Z}(E))$ is a $W^*$-correspondence.
Definitions

Center of a $W^*$-correspondence

$\mathcal{Z}(E) := \{\xi \in E \mid a \cdot \xi = \xi \cdot a \quad \forall a \in M\}$

If $(M, E)$ is a $W^*$-correspondence, then $(\mathcal{Z}(M), \mathcal{Z}(E))$ is a $W^*$-correspondence.

Isomorphism of $W^*$-correspondences

An isomorphism from $(M_1, E_1)$ to $(M_2, E_2)$ is a pair $(\sigma, \Psi)$ such that

- $\sigma : M_1 \rightarrow M_2$ is an isomorphism of $W^*$-algebras
- $\Psi : E_1 \rightarrow E_2$ is a vector space isomorphism
- for all $e, f \in E_1$ and $a, b \in M_1$,
  $\Psi(a \cdot e \cdot b) = \sigma(a) \cdot \Psi(e) \cdot \sigma(b)$ and $\langle \Psi(e), \Psi(f) \rangle = \sigma(\langle e, f \rangle)$.
Isomorphic centers

Define $\gamma : \mathcal{Z}(E) \to \mathcal{Z}(E^\sigma)$ by $\gamma(\xi) = L_\xi$, where $L_\xi : H \to E \otimes_\sigma H$ is given by $L_\xi(h) = \xi \otimes h$.

Proposition (Muhly-Solel 2008)

The pair $(\sigma, \gamma)$ is an isomorphism of $(\mathcal{Z}(M), \mathcal{Z}(E))$ onto $(\mathcal{Z}(\sigma(M)'), \mathcal{Z}(E^\sigma))$. 
Isomorphic centers

Define \( \gamma : \mathcal{Z}(E) \to \mathcal{Z}(E^\sigma) \) by \( \gamma(\xi) = L_\xi \), where \( L_\xi : H \to E \otimes_\sigma H \) is given by \( L_\xi(h) = \xi \otimes h \).

**Proposition (Muhly-Solel 2008)**

The pair \((\sigma, \gamma)\) is an isomorphism of \((\mathcal{Z}(M), \mathcal{Z}(E))\) onto \((\mathcal{Z}(\sigma(M)'), \mathcal{Z}(E^\sigma))\).

**Proposition**

The map defined on the generators of \( H^\infty(\mathcal{Z}(E)) \) by

\[
T_\xi \mapsto T_{\gamma(\xi)}, \quad \xi \in \mathcal{Z}(E) \\
\varphi_\infty(a) \mapsto \varphi_\infty^\sigma(\sigma(a)), \quad a \in \mathcal{Z}(M)
\]

extends to an isomorphism \( \Gamma \) from \( H^\infty(\mathcal{Z}(E)) \) onto \( H^\infty(\mathcal{Z}(E^\sigma)) \).
Comparison theorem revisited

Theorem (N. 2017)

Let $\zeta_1, \ldots, \zeta_N$ be $N$ distinct elements of $\mathcal{Z}(E^\sigma)$ with $\|\zeta_i\| < 1$ for all $i$, and let $\Lambda_1, \ldots, \Lambda_N \in \mathcal{Z}(\sigma(M)'$). The following are equivalent:

1. There exists $Y \in H^\infty(\mathcal{Z}(E))$ with $\|Y\| \leq 1$ such that
   \[ \hat{Y}(\zeta_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N \]
   in the sense of (Muhly-Solel 2004).

2. There exists $X = \Gamma(Y) \in H^\infty(\mathcal{Z}(E^\sigma))$ with $\|X\| \leq 1$ such that
   \[ \hat{X}(\zeta_i) = \Lambda_i, \quad i = 1, \ldots, N \]
   in the sense of (N.).
Reflections on the displacement equation approach

The displacement equation approach . . .

- avoids commutant lifting.
- can be used to recover the nontangential version of Popescu’s theorem.
- does not capture all the information in Muhly-Solel’s theorem.
- does not extend well to left-tangential Nevanlinna-Pick theorems.
References


References II


