Pick interpolation and the displacement equation

Rachael M. Norton

University of Iowa

Multivariable Operator Theory at the Technion
On the occasion of Baruch Solel's 65th birthday

Classical Nevanlinna-Pick theorem

Let $H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is bounded and analytic}\}$.

Theorem (Pick 1915)

Given N distinct points $z_1, \ldots, z_N \in \mathbb{D}$ and N points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, there exists $f \in H^\infty(\mathbb{D})$ such that $||f||_\infty \leq 1$ and

$$f(z_i) = \lambda_i, \quad i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\left[\frac{1-\overline{\lambda_i}\lambda_j}{1-\overline{z_i}z_j}\right]_{i,j=1}^N$$

is positive semidefinite.

Early generalizations

- (Nagy-Koranyi 1956) $\lambda_i \in M_n(\mathbb{C})$.
- (Sarason 1967) Commutant lifting in $H^\infty(\mathbb{D})$ implies classical Nevanlinna-Pick theorem and Nagy-Koranyi theorem.
- (Ball-Gohberg 1985) Commutant lifting in the set of block upper triangular matrices implies Nevanlinna-Pick theorem for $z_i \in M_n(\mathbb{C})$ and $\lambda_i \in M_m(\mathbb{C})$.

Generalizations of interest

Two main strategies for proving generalized noncommutative Nevanlinna-Pick theorems since 1967:

- displacement equation
- commutant lifting

Generalizations of interest

Two main strategies for proving generalized noncommutative Nevanlinna-Pick theorems since 1967:

- displacement equation
- commutant lifting

Goal:

- Understand the relationship between these two approaches

Generalizations of interest

Two main strategies for proving generalized noncommutative Nevanlinna-Pick theorems since 1967:

- displacement equation (Constantinescu-Johnson 2003)
- commutant lifting (Muhly-Solel 2004, Popescu 2003)

Goal:

- Understand the relationship between these two approaches

Outline

Definitions

W^*-algebra

A W^*-**algebra** M is a C^*-algebra that is a dual space.

Definitions

W^*-algebra

A W^*-**algebra** M is a C^*-algebra that is a dual space.

W^*-correspondence

A W^*-**correspondence** E over a W^*-algebra M is

- a Hilbert C^*-module over M
- self-dual
- equipped with a faithful, normal $*$-homomorphism $\varphi : M \to \mathscr{L}(E)$ that gives the left action of M on E.

Examples of W^*-correspondences

- $M = E = \mathbb{C}$
  - $a \cdot c \cdot b = acb$
  - $\langle c, d \rangle = \overline{c}d$
- $M = \mathbb{C},\ E = \mathbb{C}^n$
  - $a \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \cdot b = \begin{bmatrix} ac_1 b \\ \vdots \\ ac_n b \end{bmatrix}$
  - $\left\langle \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \right\rangle = \sum \overline{c_i} d_i$

Examples cont.

- $G = (G^0, G^1, r, s)$, $M = C(G^0)$, $E = C(G^1)$
  - $(a \cdot \xi \cdot b)(e) = a(r(e))\xi(e)b(s(e))$
  - $\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$

W^*-correspondence setting

Given

- M, a W*-algebra
- E, a W*-correspondence over M

define

- the **Fock space** $\mathscr{F}(E)$ to be the ultraweak direct sum $\bigoplus_{k=0}^{\infty} E^{\otimes k}$, where $E^{\otimes 0} = M$, viewed as a bimodule over itself
- the von Neumann algebra of bounded operators $\mathscr{L}(\mathscr{F}(E))$ on the Fock space of E

Operators on the Fock space $\mathscr{F}(E)$

Define the **left action operator** $\varphi_\infty : M \to \mathscr{L}(\mathscr{F}(E))$ by

$$\varphi_\infty(a) = \begin{bmatrix} a & & & \\ & \varphi(a) & & \\ & & \varphi_2(a) & \\ & & & \ddots \end{bmatrix}$$

where $\varphi_k(a) : E^{\otimes k} \to E^{\otimes k}$ is given by

$$\varphi_k(a)(\xi_1 \otimes \xi_2 \otimes \dots \xi_k) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \dots \xi_k.$$

Operators on the Fock space $\mathscr{F}(E)$ cont.

For $\xi \in E$, define the **left creation operator** $T_\xi : \mathscr{F}(E) \to \mathscr{F}(E)$ by $T_\xi(\eta) = \xi \otimes \eta$, i.e.,

$$T_\xi = \begin{bmatrix} 0 & & & \\ T_\xi^{(1)} & 0 & & \\ & T_\xi^{(2)} & 0 & \\ & & \ddots & \ddots \end{bmatrix}$$

where $T_\xi^{(k)} : E^{\otimes k-1} \to E^{\otimes k}$ is given by

$$T_\xi^{(k)}(\eta_1 \otimes \ldots \otimes \eta_{k-1}) = \xi \otimes \eta_1 \otimes \ldots \otimes \eta_{k-1}.$$

Subalgebras of $\mathscr{L}(\mathscr{F}(E))$

Tensor algebra of E

The **tensor algebra** of E, denoted $\mathscr{T}_+(E)$, is the norm-closed subalgebra of $\mathscr{L}(\mathscr{F}(E))$ generated by $\{\varphi_\infty(a) \mid a \in M\}$ and $\{T_\xi \mid \xi \in E\}$.

Subalgebras of $\mathscr{L}(\mathscr{F}(E))$

Tensor algebra of E

The **tensor algebra** of E, denoted $\mathscr{T}_+(E)$, is the norm-closed subalgebra of $\mathscr{L}(\mathscr{F}(E))$ generated by $\{\varphi_\infty(a) \mid a \in M\}$ and $\{T_\xi \mid \xi \in E\}$.

Hardy algebra of E

The **Hardy algebra** of E, denoted $H^\infty(E)$, is the ultraweak closure of $\mathscr{T}_+(E)$ in $\mathscr{L}(\mathscr{F}(E))$.

The σ-dual E^σ

Given

- M, a W^*-algebra
- E, a W^*-correspondence
- $\sigma : M \to B(H)$, a faithful, normal representation of M on a Hilbert space H,

define

- $E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I_H)\eta \, \forall a \in M\}$.

E^σ is a W^*-correspondence over $\sigma(M)'$

$$E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I_H)\eta \, \forall a \in M\}$$

E^σ is a W^*-correspondence over $\sigma(M)'$:

- $a \cdot \eta \cdot b := (I_E \otimes a)\eta b$
- $\langle \eta, \xi \rangle := \eta^* \xi$

Construct $H^\infty(E^\sigma)$.

Cauchy kernel

$$E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I_H)\eta \, \forall a \in M\}$$

For $\eta \in E^\sigma$ with $\|\eta\| < 1$ and $k \in \mathbb{N}$, define

- the *kth* **tensorial power** $\eta^{(k)} \in B(H, E^{\otimes k} \otimes_\sigma H)$ by

$$\eta^{(k)} = (I_{E^{\otimes k-1}} \otimes \eta)(I_{E^{\otimes k-2}} \otimes \eta) \cdots (I_E \otimes \eta)\eta$$

- the **Cauchy kernel** $C(\eta) \in B(H, \mathscr{F}(E) \otimes_\sigma H)$ by

$$C(\eta) = \begin{bmatrix} I_H & \eta & \eta^{(2)} & \eta^{(3)} & \dots \end{bmatrix}^T$$

Spectral radius in the W^*-correspondence setting

Spectral radius

For $\eta \in E^\sigma$, define the **spectral radius** of η by

$$r(\eta) := \inf_k \|\eta^{(k)}\|^{1/k}.$$

Spectral radius in the W^*-correspondence setting

Spectral radius

For $\eta \in E^\sigma$, define the **spectral radius** of η by

$$r(\eta) := \inf_k \|\eta^{(k)}\|^{1/k}.$$

Proposition

For $\eta \in E^\sigma$, $C(\eta) \in B(H, \mathscr{F}(E) \otimes_\sigma H)$ if and only if $r(\eta) < 1$.

Spectral radius in the W^*-correspondence setting

Spectral radius

For $\eta \in E^\sigma$, define the **spectral radius** of η by

$$r(\eta) := \inf_k \|\eta^{(k)}\|^{1/k}.$$

Proposition

For $\eta \in E^\sigma$, $C(\eta) \in B(H, \mathscr{F}(E) \otimes_\sigma H)$ if and only if $r(\eta) < 1$.

For $\eta \in E^\sigma$, $\|\eta\| < 1$ implies $r(\eta) < 1$, but the converse is not true.

Point evaluation

Define $U : \mathscr{F}(E^\sigma) \otimes_\iota H \to \mathscr{F}(E) \otimes_\sigma H$ by

$$U(\eta_1 \otimes \cdots \otimes \eta_k \otimes h) = (I_{E^{\otimes k-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{k-1}) \eta_k h.$$

Define $\rho : H^\infty(E^\sigma) \to B(\mathscr{F}(E) \otimes_\sigma H)$ by

$$\rho(X) = U(X \otimes I_H) U^*.$$

Point evaluation

Define $U : \mathscr{F}(E^\sigma) \otimes_\iota H \to \mathscr{F}(E) \otimes_\sigma H$ by

$$U(\eta_1 \otimes \cdots \otimes \eta_k \otimes h) = (I_{E^{\otimes k-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{k-1})\eta_k h.$$

Define $\rho : H^\infty(E^\sigma) \to B(\mathscr{F}(E) \otimes_\sigma H)$ by

$$\rho(X) = U(X \otimes I_H)U^*.$$

Point evaluation

For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $r(\eta) < 1$, define the **point evaluation** $\hat{X}(\eta)$ by

$$\hat{X}(\eta) = C(0)^* \rho(X)^* C(\eta).$$

Remarks about the point evaluation

Point evaluation

For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $r(\eta) < 1$, define the **point evaluation** $\hat{X}(\eta)$ by

$$\hat{X}(\eta) = C(0)^* \rho(X)^* C(\eta).$$

- $\hat{X}(\eta) \in \sigma(M)'$

Remarks about the point evaluation

Point evaluation

For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $r(\eta) < 1$, define the **point evaluation** $\hat{X}(\eta)$ by

$$\hat{X}(\eta) = C(0)^* \rho(X)^* C(\eta).$$

- $\hat{X}(\eta) \in \sigma(M)'$
- Not multiplicative, i.e., $\widehat{XY}(\eta) \neq \hat{X}(\eta)\hat{Y}(\eta)$

Remarks about the point evaluation

Point evaluation

For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $r(\eta) < 1$, define the **point evaluation** $\hat{X}(\eta)$ by

$$\hat{X}(\eta) = C(0)^* \rho(X)^* C(\eta).$$

- $\hat{X}(\eta) \in \sigma(M)'$
- Not multiplicative, i.e., $\widehat{XY}(\eta) \neq \hat{X}(\eta)\hat{Y}(\eta)$
- Induces an algebra antihomomorphism from $H^\infty(E^\sigma)$ into the completely bounded maps on $\sigma(M)'$

Muhly-Solel point evaluation

Muhly-Solel point evaluation (Muhly-Solel 2004)

For $Y \in H^\infty(E)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, define the **point evaluation** $\hat{Y}(\eta^*)$ by

$$\hat{Y}(\eta^*) = \left(C(0)^*(Y^* \otimes I_H)C(\eta)\right)^* .$$

Muhly-Solel point evaluation

Muhly-Solel point evaluation (Muhly-Solel 2004)

For $Y \in H^\infty(E)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, define the **point evaluation** $\hat{Y}(\eta^*)$ by

$$\hat{Y}(\eta^*) = (C(0)^*(Y^* \otimes I_H)C(\eta))^* .$$

- $\hat{Y}(\eta^*) \in B(H)$

Muhly-Solel point evaluation

Muhly-Solel point evaluation (Muhly-Solel 2004)

For $Y \in H^\infty(E)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, define the **point evaluation** $\hat{Y}(\eta^*)$ by

$$\hat{Y}(\eta^*) = (C(0)^*(Y^* \otimes I_H)C(\eta))^* .$$

- $\hat{Y}(\eta^*) \in B(H)$
- Multiplicative, i.e., $\widehat{XY}(\eta^*) = \hat{X}(\eta^*)\hat{Y}(\eta^*)$

Generalized Nevanlinna-Pick theorem

Theorem (N.)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of E^σ with $r(\mathfrak{z}_i) < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in \sigma(M)'$. There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that

$$\hat{X}(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the operator matrix

$$\mathcal{A}_\mathcal{N} = \left[C(\mathfrak{z}_i)^*(I_{\mathscr{F}(E)} \otimes (I_H - \Lambda_i^*\Lambda_j))C(\mathfrak{z}_j) \right]_{i,j=1}^N$$

is positive semidefinite.

Corollary: Classical Nevanlinna-Pick theorem

If

- $M = E = \mathbb{C}$
- $\sigma : M \to B(\mathbb{C})$ is given by $\sigma(a) = a$

then

- $E^\sigma = \mathbb{C}$
- $\sigma(M)' = \mathbb{C}$
- we recover the classical Nevanlinna-Pick theorem

Corollary: Constantinescu-Johnson's theorem

If

- $M = \mathbb{C}, E = \mathbb{C}^n$
- $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$

then

- $E^\sigma = C_n(B(H))$
- $\sigma(M)' = B(H)$
- we recover Constantinescu-Johnson's theorem

Displacement equation

A **displacement equation** is an equation of the form

$$(I_{B(H)} - \theta)(A) = B,$$

where $A, B \in B(H)$ and $\theta : B(H) \to B(H)$.

Displacement equation

A **displacement equation** is an equation of the form

$$(I_{B(H)} - \theta)(A) = B,$$

where $A, B \in B(H)$ and $\theta : B(H) \to B(H)$.
Given θ and B, and assuming $(I_{B(H)} - \theta)^{-1}$ exists, solve for A:

$$A = (I_{B(H)} - \theta)^{-1}(B) = \sum_{k=0}^{\infty} \theta^k(B).$$

Displacement equation

A **displacement equation** is an equation of the form

$$(I_{B(H)} - \theta)(A) = B,$$

where $A, B \in B(H)$ and $\theta : B(H) \to B(H)$.
Given θ and B, and assuming $(I_{B(H)} - \theta)^{-1}$ exists, solve for A:

$$A = (I_{B(H)} - \theta)^{-1}(B) = \sum_{k=0}^{\infty} \theta^k(B).$$

We are interested in the case when θ is completely positive. In this case, $(I_{B(H)} - \theta)^{-1}$ is completely positive as well.

Proof of (N.)

Step 1

Let $\mathfrak{z} = \begin{bmatrix} \mathfrak{z}_1 & & \\ & \ddots & \\ & & \mathfrak{z}_N \end{bmatrix}$, $U = \begin{bmatrix} I_H \\ \vdots \\ I_H \end{bmatrix}$, and $V = \begin{bmatrix} \Lambda_1^* \\ \vdots \\ \Lambda_N^* \end{bmatrix}$. Form the displacement equation

$$(I_{B(H)} - \theta_{\mathfrak{z}})(A) = UU^* - VV^*,$$

where $A \in B(H)$ and $\theta_{\mathfrak{z}}(A) = \mathfrak{z}^*(I_E \otimes A)\mathfrak{z}$.

Proof cont.

Step 2

Observe

- The Pick matrix is the unique solution of the displacement equation, i.e.,

$$A = \mathcal{A}_{\mathcal{N}}$$

Proof cont.

Step 2

Observe

- The Pick matrix is the unique solution of the displacement equation, i.e.,
$$A = \mathcal{A}_{\mathcal{N}}$$
- We can rewrite the Pick matrix as $\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty$, where $U_\infty = \begin{bmatrix} C(\mathfrak{z}_1) & \cdots & C(\mathfrak{z}_N) \end{bmatrix}$ and $V_\infty = \begin{bmatrix} (I_{\mathscr{F}(E)} \otimes \Lambda_1)C(\mathfrak{z}_1) & \cdots & (I_{\mathscr{F}(E)} \otimes \Lambda_N)C(\mathfrak{z}_N) \end{bmatrix}$.

Proof cont.

Lemma (Step 3)

$\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty$ is positive semidefinite if and only if there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $\rho(X)^* U_\infty = V_\infty$.

Proof cont.

Lemma (Step 3)

$\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty$ is positive semidefinite if and only if there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $\rho(X)^* U_\infty = V_\infty$.

Step 4

$\rho(X)^* U_\infty = V_\infty$ if and only if $\hat{X}(\mathfrak{z}_i) = \Lambda_i$ for all i.

Proof of lemma

Lemma

$\mathcal{A}_{\mathcal{N}} = U_\infty^* U_\infty - V_\infty^* V_\infty$ is positive semidefinite if and only if there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $\rho(X)^* U_\infty = V_\infty$.

Proof: $(\implies)$

- $\mathcal{A}_{\mathcal{N}} \geq 0 \implies \exists L \in \sigma^{(N)}(M)'$ such that $\mathcal{A}_{\mathcal{N}} = LL^*$
- Displacement equation becomes $\hat{A}^* \hat{A} = \hat{B}^* \hat{B}$

Proof of lemma cont.

- Douglas's lemma $\implies \exists!$ partial isometry Ω such that
  - $\hat{A} = \Omega \hat{B}$
  - $Inn(\Omega) \subseteq \overline{Range(\hat{B})}$
- Define matrix T in terms of the entries of Ω so that $TU_\infty = V_\infty$.
- There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $T = \rho(X)^*$, and $\rho(X)^* U_\infty = V_\infty$.

Proof of lemma cont.

($\Longleftarrow$) If there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\rho(X)^* U_\infty = V_\infty$, then

$$\begin{aligned} \mathcal{A}_\mathcal{N} &= U_\infty^* U_\infty - V_\infty^* V_\infty \\ &= U_\infty^* U_\infty - U_\infty^* \rho(X)\rho(X)^* U_\infty \\ &= U_\infty^* (I - \rho(X)\rho(X)^*) U_\infty \\ &\geq 0 \end{aligned}$$

since $\|X\| \leq 1$ and ρ is an isometry.

Popescu's setting

- $M = \mathbb{C}$
- $E = \mathbb{C}^n$
- $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$
- $E^\sigma = C_n(B(H))$
- $\sigma(M)' = B(H)$

Popescu's setting

- $M = \mathbb{C}$
- $E = \mathbb{C}^n$
- $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$
- $E^\sigma = C_n(B(H))$
- $\sigma(M)' = B(H)$

Spectral radius

For $\mathfrak{z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix} \in B(H)^n$, the **spectral radius** of $\mathfrak{z}$ is given by

$$r(\mathfrak{z}) := \inf_k \left\| \sum_{|\alpha|=k} Z_\alpha (Z_\alpha)^* \right\|^{1/2k}$$

Popescu's setting

- $M = \mathbb{C}$
- $E = \mathbb{C}^n$
- $\sigma : M \to B(H)$ is given by $\sigma(a) = aI_H$
- $E^\sigma = C_n(B(H))$
- $\sigma(M)' = B(H)$

Spectral radius

For $\mathfrak{z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix} \in B(H)^n$, the **spectral radius** of $\mathfrak{z}$ is given by

$$r(\mathfrak{z}) := \inf_k \left\| \sum_{|\alpha|=k} Z_\alpha (Z_\alpha)^* \right\|^{1/2k} = \inf_k \|\mathfrak{z}^{*(k)}\|^{1/k} = r(\mathfrak{z}^*).$$

Popescu's setting cont.

Let $S_1, \ldots, S_n$ be the **left creation operators** on the Fock space of $\mathbb{C}^n$. For $\Phi \in H^\infty(\mathbb{C}^n)\overline{\otimes}B(H)$, we can write

$$\Phi = \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)}, \quad A_{(\alpha)} \in B(H).$$

Point evaluation

Define the **point evaluation** of $\Phi = \sum_{\alpha \in F_n^+} S_\alpha \otimes A_{(\alpha)}$ at $\mathfrak{z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix}$ with $r(\mathfrak{z}) < 1$ by

$$\Phi(\mathfrak{z}) := \sum_{\alpha \in F_n^+} Z_{\widetilde{\alpha}} A_{(\alpha)},$$

where $\widetilde{\alpha}$ denotes the reverse of α.

Nontangential version of Popescu's theorem

Theorem (Popescu 2003)

For $i = 1, \ldots, N$, let $\mathfrak{z}_i = \begin{bmatrix} Z_{i1} & \cdots & Z_{in} \end{bmatrix} \in B(H)^n$ with $r(\mathfrak{z}_i) < 1$, and let $\Lambda_i \in B(H)$. There exists $\Phi \in H^\infty(\mathbb{C}^n)\overline{\otimes}B(H)$ such that $\|\Phi\| \leq 1$ and

$$\Phi(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the operator matrix

$$\mathcal{A}_{\mathcal{P}} = \left[\sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_{i\alpha}(I_H - \Lambda_i \Lambda_j^*)(Z_{j\alpha})^*\right]_{i,j=1}^{N}$$

is positive semidefinite.

Proof via the displacement equation

1. Given the data in Popescu's theorem, (N.) implies that there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ for all i if and only if the Pick matrix $\mathcal{A}_\mathcal{N}$ is positive semidefinite.

Proof via the displacement equation

1. Given the data in Popescu's theorem, (N.) implies that there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ for all i if and only if the Pick matrix $\mathcal{A}_\mathcal{N}$ is positive semidefinite.
2. The Pick matrices $\mathcal{A}_\mathcal{P}$ and $\mathcal{A}_\mathcal{N}$ are equal.

Proof via the displacement equation

1. Given the data in Popescu's theorem, (N.) implies that there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ for all i if and only if the Pick matrix $\mathcal{A}_\mathcal{N}$ is positive semidefinite.
2. The Pick matrices $\mathcal{A}_\mathcal{P}$ and $\mathcal{A}_\mathcal{N}$ are equal.
3. There exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ if and only if there exists $\Phi \in H^\infty(\mathbb{C}^n)\overline{\otimes}B(H)$ such that $\|\Phi\| \leq 1$ and $\Phi(\mathfrak{z}_i) = \Lambda_i$.

Proof via the displacement equation

1. Given the data in Popescu's theorem, (N.) implies that there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ for all i if and only if the Pick matrix $\mathcal{A}_\mathcal{N}$ is positive semidefinite.
2. The Pick matrices $\mathcal{A}_\mathcal{P}$ and $\mathcal{A}_\mathcal{N}$ are equal.
3. There exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\hat{X}(\mathfrak{z}_i^*) = \Lambda_i^*$ if and only if there exists $\Phi \in H^\infty(\mathbb{C}^n)\overline{\otimes}B(H)$ such that $\|\Phi\| \leq 1$ and $\Phi(\mathfrak{z}_i) = \Lambda_i$. Hint: $\Phi = (J \otimes I_H)\rho(X)(J \otimes I_H)$, where $J : \mathscr{F}(\mathbb{C}^n) \to \mathscr{F}(\mathbb{C}^n)$ is given by $J(e_{i_1} \otimes \cdots \otimes e_{i_k}) = e_{i_k} \otimes \cdots \otimes e_{i_1}$.

Muhly-Solel's theorem

Theorem (Muhly-Solel 2004)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct points of E^σ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in B(H)$. There exists $Y \in H^\infty(E)$ with $\|Y\| \leq 1$ such that

$$\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N,$$

if and only if the map from $M_N(\sigma(M)')$ to $M_N(B(H))$ defined by

$$B \mapsto (I - \Psi_\Lambda) \circ (I - \theta_\mathfrak{z})^{-1}(B)$$

is completely positive, where $\Lambda = diag[\Lambda_i]$, $\mathfrak{z} = diag[\mathfrak{z}_i]$, $\Psi_\Lambda(C) = \Lambda^* C \Lambda$, and $\theta_\mathfrak{z}(C) = \mathfrak{z}^*(I_E \otimes C)\mathfrak{z}$ for all $C \in M_N(\sigma(M)')$.

Muhly-Solel's theorem

Theorem (Muhly-Solel 2004)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct points of E^σ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in B(H)$. There exists $Y \in H^\infty(E)$ with $\|Y\| \leq 1$ such that

$$\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N,$$

if and only if the map from $M_N(\sigma(M)')$ to $M_N(B(H))$ defined by

$$B \mapsto (I - \Psi_\Lambda) \circ (I - \theta_{\mathfrak{z}})^{-1}(B)$$

is completely positive, where $\Lambda = diag[\Lambda_i]$, $\mathfrak{z} = diag[\mathfrak{z}_i]$, $\Psi_\Lambda(C) = \Lambda^* C \Lambda$, and $\theta_{\mathfrak{z}}(C) = \mathfrak{z}^*(I_E \otimes C)\mathfrak{z}$ for all $C \in M_N(\sigma(M)')$.

An implication

Suppose the Muhly-Solel Pick matrix map $(I - \Psi_\Lambda) \circ (I - \theta_{\mathfrak{z}})^{-1}$ is completely positive.

An implication

Suppose the Muhly-Solel Pick matrix map $(I - \Psi_\Lambda) \circ (I - \theta_\mathfrak{z})^{-1}$ is completely positive.
$\implies (I - \theta_\mathfrak{z})^{-1} \circ (I - \Psi_\Lambda)$ is completely positive.

An implication

Suppose the Muhly-Solel Pick matrix map $(I - \Psi_\Lambda) \circ (I - \theta_{\mathfrak{z}})^{-1}$ is completely positive.

$\implies (I - \theta_{\mathfrak{z}})^{-1} \circ (I - \Psi_\Lambda)$ is completely positive.

$\implies \mathcal{A}_\mathcal{N} = (I - \theta_{\mathfrak{z}})^{-1}(UU^* - VV^*)$

An implication

Suppose the Muhly-Solel Pick matrix map $(I - \Psi_\Lambda) \circ (I - \theta_{\mathfrak{z}})^{-1}$ is completely positive.

$\implies (I - \theta_{\mathfrak{z}})^{-1} \circ (I - \Psi_\Lambda)$ is completely positive.

$$\implies \mathcal{A}_{\mathcal{N}} = (I - \theta_{\mathfrak{z}})^{-1}(UU^* - VV^*)$$

$$= (I - \theta_{\mathfrak{z}})^{-1} \circ (I - \Psi_\Lambda) \left(\begin{bmatrix} I_H & \cdots & I_H \\ \vdots & & \vdots \\ I_H & \cdots & I_H \end{bmatrix} \right) \geq 0.$$

An implication

Suppose the Muhly-Solel Pick matrix map $(I - \Psi_\Lambda) \circ (I - \theta_\mathfrak{z})^{-1}$ is completely positive.

$\implies$ $(I - \theta_\mathfrak{z})^{-1} \circ (I - \Psi_\Lambda)$ is completely positive.

$$\implies \mathcal{A}_\mathcal{N} = (I - \theta_\mathfrak{z})^{-1}(UU^* - VV^*)$$
$$= (I - \theta_\mathfrak{z})^{-1} \circ (I - \Psi_\Lambda) \left(\begin{bmatrix} I_H & \cdots & I_H \\ \vdots & & \vdots \\ I_H & \cdots & I_H \end{bmatrix} \right) \geq 0.$$

Moral: Interpolation in the sense of Muhly-Solel's theorem implies interpolation in the sense of (N.). However, a simple example shows that the converse is not true.

Comparison theorem

Theorem (N. 2017)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of $\mathfrak{Z}(E^\sigma)$ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in \mathfrak{Z}(\sigma(M)')$.

Comparison theorem

Theorem (N. 2017)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of $\mathfrak{Z}(E^\sigma)$ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in \mathfrak{Z}(\sigma(M)')$. The following are equivalent:

1. There exists $Y \in H^\infty(\mathfrak{Z}(E))$ with $\|Y\| \leq 1$ such that
$$\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N$$
in the sense of (Muhly-Solel 2004).
2. There exists $X \in H^\infty(\mathfrak{Z}(E^\sigma))$ with $\|X\| \leq 1$ such that
$$\hat{X}(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N$$
in the sense of (N.).

Definitions

Center of a W^*-correspondence

$$\mathfrak{Z}(E) := \{\xi \in E \mid a \cdot \xi = \xi \cdot a \quad \forall a \in M\}$$

If (M, E) is a W^*-correspondence, then $(\mathfrak{Z}(M), \mathfrak{Z}(E))$ is a W^*-correspondence.

Definitions

Center of a W^*-correspondence

$$\mathfrak{Z}(E) := \{\xi \in E \mid a \cdot \xi = \xi \cdot a \quad \forall a \in M\}$$

If (M, E) is a W^*-correspondence, then $(\mathfrak{Z}(M), \mathfrak{Z}(E))$ is a W^*-correspondence.

Isomorphism of W^*-correspondences

An **isomorphism** from (M_1, E_1) to (M_2, E_2) is a pair (σ, Ψ) such that

- $\sigma : M_1 \to M_2$ is an isomorphism of W^*-algebras
- $\Psi : E_1 \to E_2$ is a vector space isomorphism
- for all $e, f \in E_1$ and $a, b \in M_1$,
  $\Psi(a \cdot e \cdot b) = \sigma(a) \cdot \Psi(e) \cdot \sigma(b)$ and $\langle \Psi(e), \Psi(f) \rangle = \sigma(\langle e, f \rangle)$.

Isomorphic centers

Define $\gamma : \mathfrak{Z}(E) \to \mathfrak{Z}(E^\sigma)$ by $\gamma(\xi) = L_\xi$, where $L_\xi : H \to E \otimes_\sigma H$ is given by $L_\xi(h) = \xi \otimes h$.

Proposition (Muhly-Solel 2008)

The pair (σ, γ) is an isomorphism of $(\mathfrak{Z}(M), \mathfrak{Z}(E))$ onto $(\mathfrak{Z}(\sigma(M)'), \mathfrak{Z}(E^\sigma))$.

Isomorphic centers

Define $\gamma : \mathfrak{Z}(E) \to \mathfrak{Z}(E^\sigma)$ by $\gamma(\xi) = L_\xi$, where $L_\xi : H \to E \otimes_\sigma H$ is given by $L_\xi(h) = \xi \otimes h$.

Proposition (Muhly-Solel 2008)

The pair (σ, γ) is an isomorphism of $(\mathfrak{Z}(M), \mathfrak{Z}(E))$ onto $(\mathfrak{Z}(\sigma(M)'), \mathfrak{Z}(E^\sigma))$.

Proposition

The map defined on the generators of $H^\infty(\mathfrak{Z}(E))$ by

$$\begin{aligned} T_\xi &\mapsto T_{\gamma(\xi)}, \quad \xi \in \mathfrak{Z}(E) \\ \varphi_\infty(a) &\mapsto \varphi_\infty^\sigma(\sigma(a)), \quad a \in \mathfrak{Z}(M) \end{aligned}$$

extends to an isomorphism Γ from $H^\infty(\mathfrak{Z}(E))$ onto $H^\infty(\mathfrak{Z}(E^\sigma))$.

Comparison theorem revisited

Theorem (N. 2017)

Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_N$ be N distinct elements of $\mathfrak{Z}(E^\sigma)$ with $\|\mathfrak{z}_i\| < 1$ for all i, and let $\Lambda_1, \ldots, \Lambda_N \in \mathfrak{Z}(\sigma(M)')$. The following are equivalent:

1. There exists $Y \in H^\infty(\mathfrak{Z}(E))$ with $\|Y\| \leq 1$ such that
$$\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N$$
in the sense of (Muhly-Solel 2004).
2. There exists $X = \Gamma(Y) \in H^\infty(\mathfrak{Z}(E^\sigma))$ with $\|X\| \leq 1$ such that
$$\hat{X}(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N$$
in the sense of (N.).

Reflections on the displacement equation approach

The displacement equation approach . . .

- avoids commutant lifting.
- can be used to recover the nontangential version of Popescu's theorem.
- does not capture all the information in Muhly-Solel's theorem.
- does not extend well to left-tangential Nevanlinna-Pick theorems.

References I

J. Agler and J. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, Amer. Math. Soc., Providence, RI (2002).

T. Constantinescu and J. L. Johnson, *A note on noncommutative interpolation*, Canad. Math. Bull. **46** (1) (2003), 59-70.

R. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413-415.

J. Good, *Interpolation and commutant lifting with weights*, Integr. Equ. Oper. Theory **87** (2017), 349-389.

References II

- P. Muhly and B. Solel, *Hardy algebras, W^*-correspondences and interpolation theory*, Math. Ann. **330** (2004), 353-415.
- P. Muhly and B. Solel, *The Poisson kernel for Hardy algebras*, Complex Anal. Oper. Theory **3** (2009), 221-242.
- R. Norton, *Comparing two generalized noncommutative Nevanlinna-Pick theorems*, Complex Anal. Oper. Theory **11** (2017), 875-894.
- G. Popescu, *Multivariable Nehari problem and interpolation*, J. Funct. Anal. **200** (2003), 536-581.