Matrix Bundles in Free Analysis: Where Free Functions Live and Thrive

Based on joint work with E. Griesenauer and B. Solel

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MVOT
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And special thanks to Baruch

Happy Birthday Baruch

Motivation / Inspiration

- Arveson's theory of subalgebras of C^*-algebras. In particular, his theory of boundary representations.
- *Foundations of Free Noncommutative Function Theory* by D. Kaliuzhnyi-Verbovetskyi and V. Vinnikov.
- Groucho Marx
- Hermann Weyl

Motivation: Arveson's Boundary Theory

- $1 \in A \subseteq B$ - a unital not-necessarily self-adjoint subalgebra A of a C^*-algebra B (same unit).
- $C^*(A) :=$ the C^*-subalgebra of B generated by A.

Definition

A **boundary representation** *of $C^*(A)$ for A is an irreducible C^*-representation $\pi : C^*(A) \to B(H_\pi)$ with the property that the only completely positive map $\varphi : C^*(A) \to B(H_\pi)$ such that $\varphi|A = \pi|A$ is π.*

- $\partial Rep(C^*(A))$ – all boundary reps of $C^*(A)$ for A.

Motivation: Arveson's Boundary Theory

- $\bigcap\{\ker(\pi) \mid \pi \in \partial Rep(C^*(A))\} := \mathfrak{S}(C^*(A))$ – the **Shilov boundary ideal** of $C^*(A)$.
- $C^*(A)/\mathfrak{S}(C^*(A)) := C_e^*(A)$ – the C^*-envelope.

Theorem (Arveson)

The quotient map $q : C^*(A) \to C_e^*(A)$ is completely isometric on A and allows us to view $A \subseteq C_e^*(A)$. Every completely isometric isomorphism between operator algebras A_1 and A_2 is the restriction of a C^*-isomorphism from $C_e^*(A_1)$ to $C_e^*(A_2)$.

The Basic Example

Example

- $A(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$, the disc algebra.
- $C^*(A(\mathbb{D})) = C(\overline{\mathbb{D}})$
- $\partial Rep(A(\mathbb{D})) = \mathbb{T}$ the boundary of $\overline{\mathbb{D}}$.
- $C_e^*(A(\mathbb{D})) = C(\mathbb{T})$

Motivation from Free Analysis

Definitions

- M_n^d - d-tuples of $n \times n$ complex matrices.
- For $\mathfrak{z} \in M_n^d$, write $\mathfrak{z} := (Z_1, Z_2, \cdots Z_d)$, $Z_i \in M_n$ and set $\mathcal{Z}_k : M_n^d \to M_n$, $\mathcal{Z}_k(\mathfrak{z}) = Z_k$ - the k^{th}-matricial coordinate function.
- $\mathbb{G}_0 = \mathbb{G}_0(d, n) :=$ the algebra of M_n-valued functions on M_n^d generated by $\{\mathcal{Z}_k\}_{k=1}^d$ in $M_n(\mathbb{C}[M_n^d])$ – the polynomial algebra of d generic $n \times n$ matrices.
- Why these algebras? Answer: They are the algebras that arise from representing the free algebra $\mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle$ as functions on its space of n-dimensional representations.

Taylor / K-V, V Axioms

- $\mathcal{M}^d := \coprod_{n\geq 1} M_n^d$
- $\Omega = \coprod_{n\geq 1} \Omega_n \subseteq \mathcal{M}^d$ - an open N.C. set, i.e., Ω_n is open in M_n^d and $\Omega_k \oplus \Omega_l \subseteq \Omega_{k+l}$.

Definition

A function $f : \Omega \to \mathcal{M}^1$ ($f = \{f_n\}_{n\geq 1}$) is called a noncommutative function iff

1. $f_n(\Omega_n) \subseteq M_n$, *for all* n
2. $f_{k+l}(X \oplus Y) = f_k(X) \oplus f_l(Y), \quad X \in \Omega_k, \ Y \in \Omega_l.$
3. $f_n(SXS^{-1}) = Sf_n(X)S^{-1}$, *for all* $X \in \Omega_n$ *and* $S \in Gl(n, \mathbb{C})$ *such that* $SXS^{-1} \in \Omega_n$.

Amazing Fact

- Under anodyne boundedness hypotheses, a noncommutative function $f = \{f_n\}$ is analytic, i.e., each f_n is analytic.

Motivation from Groucho Marx

- Recall axioms 2 and 3 for noncommutative functions:

  (2) $f_{k+l}(X \oplus Y) = f_k(X) \oplus f_l(Y), \quad X \in \Omega_k,\ Y \in \Omega_l.$

  (3) $f_n(SXS^{-1}) = Sf_n(X)S^{-1}$, for all $X \in \Omega_n$ and $S \in Gl(n, \mathbb{C})$ such that $SXS^{-1} \in \Omega_n$.

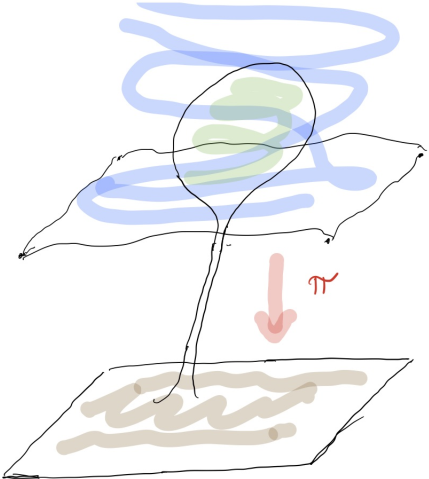
- Taylor observed that these two axioms can be replaced by this single one: For $X \in \Omega_n$, $Y \in \Omega_m$, and $T \in M_{n\times m}$ such that $XT = TY$, $f_n(X)T = Tf_m(Y)$, where if $X = (X_1, X_2, \cdots, X_d)$, then $XT = (X_1T, X_2T, \cdots, X_dT)$, similarly for TY.

- My reaction: "Anyone who says he can see through women is missing a lot." (Groucho Marx)

Motivation from Hermann Weyl

- "I shared his [Felix Klein's] conviction that Riemann surfaces are not merely a device for visualizing the many-valuedness of analytic functions, but rather an indispensable essential component of the theory; not a supplement, more or less artificially distilled from the functions, but their native land, the only soil in which the functions grow and thrive."
- Reflection on the axiom $f_n(SXS^{-1}) = Sf_n(X)S^{-1}$, for all $X \in \Omega_n$ and $S \in Gl(n, \mathbb{C})$ such that $SXS^{-1} \in \Omega_n$ reveals that Ω_n is not the natural domain for f_n.

A Thriving Free Function at Home



An Old Idea

- R and S two unital rings.
- $X := \mathrm{Hom}(R, S)$ – unital homomorphisms.
- For $r \in R$, define $\widehat{r} : X \to S$ by

$$\widehat{r}(\varphi) := \varphi(r),$$

  and let $\widehat{R} := \{\widehat{r} \mid r \in R\}$.
- Clearly, $r \to \widehat{r}$ is a homomorphism (i.e., a representation) of R into $\widehat{R}$.
- What information do these functions carry?
- Dedekind u. Weber, *Theorie der algebraischen Functionen einer Veränderlichen*, Crelle **XCII**, Heft 3, 1882, s. 236.

Points of the Riemann Surface

1. *Definition.* If *all* the individual elements $\alpha, \beta, \gamma, \ldots$ of the field Ω are replaced by *certain* numerical values, $\alpha_0, \beta_0, \gamma_0 \ldots$, such that

(I.) $\alpha_0 = \alpha$, in the case where α is a constant, and in general

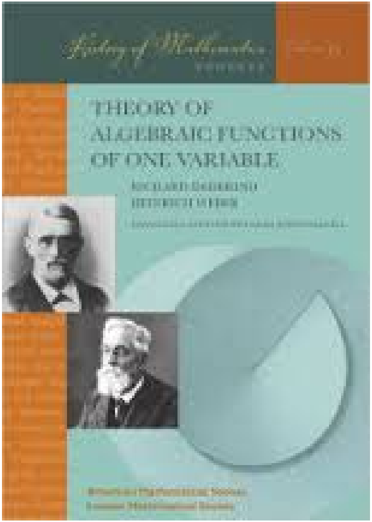
$$\text{(II.)}\quad (\alpha+\beta)_0 = \alpha_0 + \beta_0, \qquad \text{(IV.)}\quad (\alpha\beta)_0 = \alpha_0\beta_0,$$

$$\text{(III.)}\quad (\alpha-\beta)_0 = \alpha_0 - \beta_0, \qquad \text{(V.)}\quad \left(\frac{\alpha}{\beta}\right)_0 = \frac{\alpha_0}{\beta_0}$$

then a definite set of values will thus be assigned to a *point* $\mathfrak{P}$ (which one may consider for visualisation as somehow located in space [1]), and we say $\alpha = \alpha_0$ *at* $\mathfrak{P}$ or α has the value α_0 *at* $\mathfrak{P}$. Two points are called distinct if and only if there is a function α in Ω which has different values at the two points.

Translation of Dedekind and Weber by John Stillwell

History of Mathematics # 39, AMS 2012



An Old Idea Refined

- Defining $\widehat{r}: X \to S$ by $\widehat{r}(\varphi) := \varphi(r)$ may have "natural redundancies", especially if S is not commutative.
- Let $G \subseteq \mathrm{Aut}(S)$ be a subgroup, e.g. take $G = \mathrm{InnAut}(S)$. Then G acts on X as well as on S:

$$\varphi \cdot g(r) := \varphi(r) \cdot g, \qquad r \in R, \varphi \in X, g \in G.$$

- Further

$$\widehat{r}(\varphi \cdot g) \;=\; (\varphi \cdot g)(r) \;=\; \varphi(r) \cdot g \;=\; \widehat{r}(\varphi) \cdot g.$$

- Thus each $\widehat{r}$ is a "*G-concomitant*".

Concomitants

Definition

If a group G acts on two spaces X and Y, then a function $f : X \to Y$ is called a G-concomitant in case

$$f(x \cdot g) = f(x) \cdot g, \qquad g \in G, x \in X.$$

- Concomitants are also called covariants, fixed functions, invariant functions, equivariant functions, intertwiners. . . .

Observation
Tautological

Observation

Suppose G acts on X and Y. Let $\pi_0 : X \to X/G$ be the quotient map. Let $X * Y = (X \times Y)/G$ (product action), and define $\pi : X * Y \to X/G$ by $\pi([x, y]) = \pi_0(x)$. Then we are led to think of $X * Y$ as the total space of a fibre bundle over X/G with fibre Y and projection π. Further, if $f : X \to Y$ is a G-concomitant then the map $\sigma_f : X/G \to X * Y$, defined by $\sigma_f([x]) = [x, f(x)]$ is a section of this bundle, i.e. $\pi \circ \sigma_f = id_{X/G}$. Conversely, assuming the action of G on X is free (i.e., $x \cdot g = x$ implies $g = e$, for any $x \in X$), every section of this bundle is given by a concomitant.

Ineluctable Conclusion

- If one wants to study a ring R in terms of the *similarity classes* of its representations in a noncommutative ring S, one is naturally led to study $\widehat{R}$ as sections of the bundle whose base is $X/\text{InnAut}(S)$ and whose total space is $(X \times S)/\text{InnAut}(S)$.
- This observation lies at the center of the theory of moduli.
- Unfortunately, the process is sometimes not straightforward - the quotient space $X/\text{InnAut}(S)$ can be problematic.

Generic Matrices as Concomitants

- $G := PGL(n, \mathbb{C})$, viewed as the automorphism group of M_n. Write $A \cdot g := g^{-1}Ag, \quad A \in M_n, \quad g \in G$.
- G acts on M_n^d (the diagonal action): $\mathfrak{z} \cdot g := (Z_1 \cdot g, Z_2 \cdot g, \cdots, Z_d \cdot g)$.
- Since $\mathcal{Z}_k(\mathfrak{z} \cdot g) = \mathcal{Z}_k(\mathfrak{z}) \cdot g$, each $\mathcal{Z}_k$ is a G-concomitant. Thus $\mathbb{G}_0$ consists of G-concomitants.
- What does the bundle perspective have to offer for these concomitants?

Important Contributions of Procesi

- *Polynomial invariants*:

  $$\mathbb{I}_0 = \mathbb{C}[M_n^d]^G := \{p : M_d^n \xrightarrow{\text{polynomial}} \mathbb{C} \mid p(\mathfrak{z}\cdot g) = p(\mathfrak{z})\}$$

  viewed as scalar multiples of I_n.
- *Trace Algebra* $\mathbb{S}_0 :=$ subalgebra of $M_n(\mathbb{C}[M_n^d])$ generated by $\mathbb{I}_0$ and $\mathbb{G}_0$.

Important Contributions of Procesi

Theorem (Procesi)

1. $\mathbb{I}_0$ *is generated by* $\mathfrak{z} \to tr(Z^w)$, $|w| \leq 2^n - 1$, *where* $Z^w = Z_{i_1} Z_{i_2} \cdots Z_{i_{|w|}}$ *and* $w = i_1 i_2 \cdots i_{|w|}$.
2. $\mathbb{I}_0 = \mathfrak{Z}(\mathbb{S}_0)$ *and* $\mathbb{S}_0$ *is generated as a module over* $\mathbb{I}_0$ *by* $\mathcal{Z}^w$, $|w| \leq 2^{n-1}$.
3. $\mathbb{S}_0 = M_n(\mathbb{C}[M_n^d])^G$ *– all* G*-concomitants in* $M_n(\mathbb{C}[M_n^d])$.

The Bundle Perspective for Generic Matrices and Trace Algebras

- Since elements of the trace algebra are G-concomitants, $F : M_n^d \to M_n$, the base of the bundle should be M_n^d/G.
- The total space is

$$(M_n^d \times M_n)/G = \{[\mathfrak{z}, A] \mid \mathfrak{z} \in M_n^d,\ A \in M_n\},$$

which may be identified with $M_n^{(d+1)}/G$.

- The bundle projection: $\pi([\mathfrak{z}, A]) = [\mathfrak{z}]$.

M_n^d/G vs. $M_n^d//G$ Set-Theoretic vs. Categorical Quotients

- The set theoretic quotient M_n^d/G is of little use because the G-orbits in M_n^d need not be closed.

Example

$d = 1, n = 2$: The G-orbit of the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is

$$\left\{ U_1 \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} U_2 \mid \lambda \neq 0, \quad U_i \in U(2, \mathbb{C}) \right\}$$

- Every orbit closure in M_n^d contains a unique closed orbit.
- **GOAL:** Replace the bad quotient space with a useful proxy.

Categorical Quotients
Definition

Definition

Given a category $\mathcal{C}$ and an object X of $\mathcal{C}$ endowed with an action of a group G (by automorphisms of X in $\mathcal{C}$), a *categorical quotient* for the action is a pair (Y, π) where Y is an object of $\mathcal{C}$ and π is a morphism of $\mathcal{C}$ mapping X to Y such that

1. π is invariant; i.e., $\pi \circ \sigma = \pi \circ p_2$ where $\sigma : G \times X \to X$ is the given group action and p_2 is the projection of $G \times X$ onto X; and
2. (Y, π) has this universal property: any morphism $X \to Z$ satisfying 1) factors uniquely through π .

Categorical Quotients

- If a categorical quotient for G acting on X exists, it is unique up to isomorphism (in the given category) and one usually denotes any avatar by $X//G$.

Proposition

The categorical quotient $M_n^d//G$ in the category of affine G-varieties is the (abstract) affine algebraic variety $Spec\{\mathbb{C}[M_n^d]^G\}$. It may be realized concretely as an embedded algebraic variety as follows: Choose a finite set of generators $p_1, p_2, \cdots, p_e$ for $\mathbb{C}[M_n^d]^G$ and define $\mathbf{p} : M_n^d \to \mathbb{C}^e$ by $\mathbf{p}(\mathfrak{z}) := (p_1(\mathfrak{z}), p_2(\mathfrak{z}), \cdots, p_e(\mathfrak{z}))$. The image of $\mathbf{p}$ is the embedded algebraic variety, $\mathbf{V}$, that is the common zeros of the polynomial relations among the p_i.

Some Properties of $M_n^d//G$

- We identify $M_n^d//G$ with $\mathbf{V}$ and the quotient map π with $\mathbf{p}$.
- Since every orbit closure contains a unique closed orbit, $M_n//G$ can be thought of as the space of orbit closures, or the space of closed orbits.
- Artin (1969): The orbit of $\mathfrak{z} = (Z_1, \cdots, Z_d)$ is closed if and only if $\{Z_1, Z_2, \cdots, Z_d\}$ generates a semi-simple subalgebra of M_n.
- $\mathcal{V}(d, n) := \{\mathfrak{z} \in M_n^d \mid \{Z_1, Z_2, \cdots, Z_d\}$ generates $M_n\}$. $\mathcal{V}(d, n)$ is called the set of *irreducible points* in M_n^d.

M_n^d is Almost a Principal G-bundle.

Theorem (Procesi)

$\mathcal{V}(d, n)$ is the total space of a principal G bundle with bundle map $\mathbf{p}$ restricted to $\mathcal{V}(d, n)$ and base $Q_0 = Q_0(d, n) := \mathbf{p}(\mathcal{V}(d, n))$ contained in the smooth points of $\mathbf{V}$.

- $\mathcal{V}(d, n)$ is Zariski dense in M_n^d. If n or d is greater than 2, then a function holomorphic on $\mathcal{V}(d, n)$ extends to be holomorphic on M_n^d. When $d = n = 2$, $\mathcal{V}(d, n)$ is a domain of holomorphy.

An Example

From J. J. Sylvester, 1883

- $\mathfrak{z} = (Z_1, Z_2) \in \mathcal{V}(2,2)$ if and only if $\det[Z_1, Z_2] \neq 0$.
- $\mathbb{I}_0$ is generated by the following 5 functions: $\mathfrak{z} \to tr(Z_1)$, $\mathfrak{z} \to tr(Z_2)$, $\mathfrak{z} \to tr(Z_1^2)$, $\mathfrak{z} \to tr(Z_2^2)$, and $\mathfrak{z} \to tr(Z_1 Z_2)$.
- These functions are algebraically independent, so $M_2^2//G = Spec(\mathbb{I}_0)$ is $\mathbb{C}^5$.
- $\mathbb{S}_0(2,2) = \mathbb{I}_0 + \mathcal{Z}_1\mathbb{I}_0 + \mathcal{Z}_2\mathbb{I}_0 + \mathcal{Z}_1\mathcal{Z}_2\mathbb{I}_0$.

Other Categories

Theorem (Luna)

$(\mathbf{V}, \mathbf{p})$ *serves also as the categorical quotient* $(M_n^d//G, \pi)$ *for the action of* G *on* M_n^d *in the following categories:* T_1 *spaces and continuous maps,* T_2 *spaces and continuous maps, and complex analytic varieties and holomorphic maps.*

A More Concrete Model for $M_n^d//G$

- $K := PU_n$.
- $\mathcal{HN} = \mathcal{HN}(d,n) := \{\mathfrak{z} \in M_n^d \mid \sum_{k=1}^d [Z_k^*, Z_k] = 0\}$ - the hypernormal points in M_n^d.

Theorem (Kempf-Ness)

*$\mathcal{HN} = \{\mathfrak{z} \mid g \to \|\mathfrak{z} \cdot g\|_2$ achieves its infimum at $g = e\}$. ($\|\cdot\|_2 :=$ Hilbert-Schmidt norm.) **p** is a K-equivariant map of $\mathcal{HN}$ onto $M_n^d//G$, inducing a homeomorphism $\mathcal{HN}/K \simeq M_n^d//G$.*

- The Kempf-Ness theorem is an analogue of the fact that every diagonable matrix is similar to a normal matrix.

A replacement for $(M_n^d \times M_n)/G$

- For $[\mathfrak{z}] \in \mathcal{HN}/K$, let

$$\mathfrak{M}([\mathfrak{z}]) := \{[\mathfrak{z}, A] \in (\mathcal{HN} \times M_n)/K \mid A \in \{K_{\mathfrak{z}}\}'\},$$

where $K_{\mathfrak{z}}$ is the isotropy group of $\mathfrak{z}$ in K and $[\mathfrak{z}]$ is the K-orbit of $\mathfrak{z}$.

Proposition

$\{\mathfrak{M}([\mathfrak{z}])\}_{[\mathfrak{z}]\in\mathcal{HN}/K}$ has the structure of a continuous field of C^*-algebras over $\mathcal{HN}/K \simeq M_n^d//G$. A total family of continuous fields is given by $\{[\mathfrak{z}] \to [\mathfrak{z}, F(\mathfrak{z})]\}$ where F runs over all products of functions of the form $\{\mathcal{Z}_i, \mathcal{Z}_j^*\}_{i,j=1}^d$.

$\mathcal{HN}$ as a reduction of $\mathcal{V}(d, n)$

Proposition

Over $Q_0(d, n)$, $\mathcal{HN}$ is a reduction of $\mathcal{V}(d, n)$ to a principal K-bundle.

- For every compact subset $X \subseteq M_n^d//G$ (realized as $\mathcal{HN}/K$), the continuous sections of $\mathfrak{M}$ over X, $\Gamma_c(X, \mathfrak{M})$, is a C^*-algebra that is n-homogeneous when $X \subseteq Q_0(d, n)$.
- If $X \subseteq M_n^d//G$, its $\mathbb{I}_0$-*convex hull*, $\widehat{X}$, is $\{\mathfrak{z} \in M_n^d//G \mid |f(\mathfrak{z})| \leq \sup\{|f(\mathfrak{x})| \mid \mathfrak{x} \in X\},\ f \in \mathbb{I}_0\}$.
- A domain $\mathcal{D} \subseteq M_n^d//G$ is $\mathbb{I}_0$-*convex*, if $\widehat{\overline{\mathcal{D}}} = \overline{\mathcal{D}}$.

The Tracial Function Algebra $\mathbb{S}(X)$

Definition

The tracial function algebra determined by X is $\mathbb{S}(X) := \overline{\mathbb{S}_0}^{cl}$ *in* $\Gamma_c(X, \mathfrak{M})$. $\mathbb{I}(X) := \overline{\mathbb{I}_0}^{cl}$ *in* $C(X)$.

Observation

$C^*(\mathbb{S}(X)) = \Gamma_c(X, \mathfrak{M})$

Theorem (Griesenauer, M, Solel)

Let $\overline{\mathcal{D}} \subseteq Q_0(d, n)$ be $\mathbb{I}_0$-convex. Then

1. *$\overline{\mathcal{D}}$ is the maximal ideal space of $\mathbb{I}(\overline{\mathcal{D}})$.*
2. *$\mathbb{I}(\overline{\mathcal{D}})$ is the center of $\mathbb{S}(\overline{\mathcal{D}})$*
3. *$\mathbb{S}(\overline{\mathcal{D}})$ is a rank n^2-Azumaya algebra over $\mathbb{I}(\overline{\mathcal{D}})$.*

The Tracial Function Algebra $\mathbb{S}(X)$ (continued)

- If $\overline{\mathcal{D}}$ be $\mathbb{I}_0$-convex, then $\partial\mathcal{D} :=$ the Shilov boundary of $\overline{\mathcal{D}}$ viewed as the maximal ideal space of $\mathbb{I}(\overline{\mathcal{D}})$ and $\partial_e\mathcal{D}$ denotes its extreme (or Choquet) boundary.

Conjecture

If $\overline{\mathcal{D}}$ is $\mathbb{I}_0$-convex, then:

1. $\partial Rep(C^*(\mathbb{S}(\overline{\mathcal{D}}))) = \partial_e\overline{\mathcal{D}}$
2. $C_e^*(\mathbb{S}(\overline{\mathcal{D}})) = \Gamma_c(\partial\mathcal{D}, \mathfrak{M})$.

Theorem (Griesenauer, M, Solel)

If $\overline{\mathcal{D}} \subseteq Q_0(d,n)$ is $\mathbb{I}_0$-convex, then:

1. $\partial Rep(C^*(\mathbb{S}(\partial\mathcal{D}))) \supseteq \partial_e\mathcal{D}$
2. $C_e^*(\mathbb{S}(\partial\mathcal{D})) = \Gamma_c(\partial\mathcal{D}, \mathfrak{M})$.

Unfinished Business

- $\mathbb{G}_0 \subseteq \mathbb{S}_0 \subseteq \Gamma_h(\overline{\mathcal{D}}, \mathfrak{M})$. When is $\mathbb{S}_0$ dense? $\mathbb{G}_0$?
- Given a compact subset $Y \subseteq M_n^d$, let $\mathbb{S}(Y) :=$ the closure of $\mathbb{S}_0$ in $C(Y, M_n)$. When can $\mathbb{S}(Y)$ be written as $\Gamma_h(\overline{\mathcal{D}}, \mathfrak{M})$ for a suitable domain $\mathcal{D}$ in $Q_0(d, n)$? Be explicit!!
- Tell the story of $\overline{\mathbb{D}(d, n)} := \{\mathfrak{z} \in M_n^d \mid \mathfrak{z}\mathfrak{z}^* \leq I_n\}$. In particular, describe what happens on the fringe of $Q_0(d, n)$ in M_n^d?

There is a lot more I don't know, but I am out of time to discuss it.

Nevertheless. . .

Thank You