On complete K-spectral sets (the other title was too long)

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joint with Daniel Estévez and Dmitry Yakubovich

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von Neumann's Inequality: the basic version

Let p be a complex polynomial and $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space. If $\|T\| \leq 1$, then $\|p(T)\| \leq \|p\|_\infty$, where $\|p\|_\infty$ is the supremum norm of p over the unit disk $\mathbb{D}$.

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On the other hand, the Sz.-Nagy dilation theorem states that every contraction $T \in \mathcal{B}(\mathcal{H})$ has a *power dilation* $U \in \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$; that is, $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$, $n = 0, 1, 2, \ldots$.

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Consequently, we obtain the von Neumann inequality as stated.

Spectral sets

Suppose that Ω is a compact subset of $\mathbb{C}$ and $\sigma(T) \subset \Omega$. Write $R(\Omega)$ for the algebra of functions with poles off of Ω. We say that Ω is a *spectral set* for T if for all $r \in R(\Omega)$, $\|r(T)\| \leq \|r\|_\infty$, the supremum norm over Ω.

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Let $K \geq 1$. We will say that Ω is an *K-spectral set* for T if for all $r \in R(\Omega)$, $\|r(T)\| \leq K\|r\|_\infty$. So spectral sets are 1-spectral sets.

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We call Ω a *complete spectral* (*complete K-spectral set*) for T if these statements hold with $R(\Omega)$ replaced by matrix valued rational functions, so $r \in R(\Omega) \otimes M_n$ for $n \in \mathbb{N}$.

The closed disk as a complete spectral set

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The proof is essentially the same as for von Neumann’s inequality. By Runge’s theorem, we can uniformly approximate any function in $R(\overline{\mathbb{D}})$ by polynomials, and so it suffices to work with polynomials. Again the spectral theorem allows us to conclude that the result holds whenever T is a unitary operator. The Sz.-Nagy dilation theorem then gives the result for general contractions.

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Hence we have the following:

T is a contraction $\Leftrightarrow$ $\overline{\mathbb{D}}$ is a spectral set for T $\Leftrightarrow$
$\overline{\mathbb{D}}$ is a complete spectral set for T $\Leftrightarrow$ T dilates to a unitary operator.

Polynomially bounded operators and K-spectral sets

We say that an operator T is *polynomially bounded* if it has $\overline{\mathbb{D}}$ as an K-spectral set for some $K \geq 1$. Similarly it is *completely polynomially bounded* if $\overline{\mathbb{D}}$ as a complete K-spectral set.

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Suppose T is similar to a contraction; that is, there is a contraction S and an invertible operator X such that $T = X^{-1}SX$. Then one easily sees that for any polynomial p,

$$\|p(T)\| \leq \|X^{-1}\|\|X\|\|p(S)\|;$$

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Halmos' question: Is any polynomially bounded operator similar to a contraction?

Polynomial boundedness and similarity to a contraction

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There is also a nice later proof due to Davidson and Paulsen, and work by Badea extending the known counterexamples.

Further results on complete K-spectral sets

- *(Douglas-Paulsen, 1986)* : If Ω is a finitely connected domain with analytic boundary components (given by $|\varphi_k(z)| = 1$), then there is a constant K such that whenever $T \in \mathcal{B}(\mathcal{H})$ satisfies $\|\varphi_k(T)\| \leq 1$ for all k, then Ω is a complete K-spectral set for T.

  (A variation on a theorem of Arveson then implies that T is similar to an operator having a *normal rational $\partial\Omega$-dilation*; that is, T dilates to N *similar* to a normal operator with spectrum on $\partial\Omega$.)

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- *(Badea, Beckermann, Crouzeix, 2009)* : Let $\overline{\mathbb{D}_j}$ be disks in $\hat{\mathbb{C}}$, the extended complex plane, and suppose that these are 1-spectral sets for T. Then $\Omega = \bigcap D_j$ is a complete K-spectral set for T, where K depends only on the number of disks (and not T).

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- *(Mascioni, 1994)* : Let φ be a finite Blaschke product and suppose that $\overline{\mathbb{D}}$ is a complete K'-spectral set for $\varphi(T)$. Then $\overline{\mathbb{D}}$ is a complete K-spectral set for T.

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- *(Delyon, Delyon, 1999)* : Let Ω be a compact convex set containing the numerical range of an operator T. Then Ω is a complete K-spectral set for T.

Realizations and complete K-sectral sets

- Let $X \subset \hat{\mathbb{C}}$.
- Let $\Phi = \{\varphi_j\}$ be a collection of functions analytic on X (the *test functions*).
- Define $\Omega = \{z : |\varphi_j(z)| < 1\}$, $\partial\Omega = \{z : |\varphi_j(z)| = 1\}$.
- The *admissible kernels* are defined as $\mathcal{K}_\Phi = \{k \geq 0 : ((1 - \varphi_j(x)\varphi_j(y)^* k(x,y)) \geq 0\}$.
- The *Agler algebra* $H^\infty_d(\mathcal{K}_\Phi)$ of $M_d(\mathbb{C})$-valued analytic functions with norm uniformly bounded as multipliers on all $H^2(k)$, and unit ball $SA_d(\mathcal{K}_\Phi)$ (the *Schur-Agler class*).

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Theorem 1 (Realization theorem).

For $d \in \mathbb{N}$, $f : \Omega \to M_d(\mathbb{C})$, and $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \subset \Omega$, the following are equivalent:

- $f \in SA_d(\mathcal{K}_\Phi)$;
- $\|\varphi_j(T)\| < 1$ *for all* k *implies* $\|f(T)\| \leq 1$.

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- $\|\varphi_j(T)\| < 1$ *for all* k *implies* $\|f(T)\| \leq 1$*.*

In other words, any representation of $SA_1(\mathcal{K}_\Phi)$ which is strictly contractive on the test collection is completely contractive.

Our problem

Given

- a test collection $\Phi = \{\varphi_j\}$,
- $\Omega = \{z : |\varphi_j(z)| < 1\}$,
- $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \subset \overline{\Omega}$,

under what conditions is it the case that

1. If $\|\varphi_j(T)\| \leq 1$ for all j (i.e., $\overline{\mathbb{D}}$ is a complete spectral set for $\varphi_j(T)$)
   or
2. If there exists K' such that for all j, $\overline{\mathbb{D}}$ is a complete K'-spectral set for $\varphi_j(T)$

$\Rightarrow$ there exists K such that $\overline{\Omega}$ is a complete K-spectral set for T.

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In the latter case, we say that Φ is a *strong test collection*. If K does not depend on T, Φ is called a *uniform (strong) test collection*. Otherwise it is *non-uniform*.

Some observations:

- When $\sigma(T) \cap \partial\Omega \neq \emptyset$, things become more difficult!
- If $\Phi = \{\varphi\}$, a uniform test collection is automatically a strong uniform test collection, since then there exists an invertible operator S such that $S\varphi(T)S^{-1} = \varphi(STS^{-1})$ is a contraction.
- Φ is always a non-uniform strong test collection for Ω (Rota for $\mathbb{D}$, Herrero & Voiculescu in general).

Some examples of test collections

- The intersection of finitely many disks in $\widehat{\mathbb{C}}$ (Badea, Beckerman, Crouzeix) : *uniform test collection*;
- The *numerical range* of T, $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$ is a convex set which is the intersection of (generally infinitely many) closed half planes. The test collection is comprised of the associated linear functionals (Delyon-Delyon and Putinar-Sandberg) : *uniform test collection*;
- More generally, *ρ-contractions*, $T^n = \rho P_H U^n | H$, $n = 1, 2, \ldots$, the intersection of (generally infinitely many) closed disks : *uniform test collection*;
- Nice n-holed domains (Douglas-Paulsen) : *uniform strong test collection*;
- $\Phi = \{\varphi\}$, where φ is a finite Blaschke product, $\Omega = \overline{\mathbb{D}}$ (Mascioni) : *non-uniform strong test collection*;
- $\Phi = \{\varphi\}$, where φ is an infinite Blaschke product with zeros $\{\lambda_i\}$ satisfying $\sum_i (1 - |\lambda_i|^2)^{1/2} < \infty$, $\Omega = \overline{\mathbb{D}} \backslash \overline{P}$, P the poles of φ (Stessin) : *non-uniform strong test collection*;

Admissible function families

Let $\Omega \subset \mathbb{C}$ be a domain whose boundary is a disjoint finite union of piecewise analytic Jordan curves such that the interior angles of the "corners" of $\partial\Omega$ are in $(0, \pi]$. We will say that an analytic function $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is *admissible* if $\varphi_j \in \mathbb{A}(\overline{\Omega})$, for $k = 1, \ldots, n$, and there is a collection of closed analytic arcs $\{J_j\}_{k=1}^{n}$ of $\partial\Omega$ and a constant α, $0 < \alpha \leq 1$, such that the following conditions are satisfied:

(a) The arcs J_j cover all $\partial\Omega$.

(b) $|\varphi_j| = 1$ in J_j, for $k = 1, \ldots, n$.

(c) For each j, $\ldots$ and φ_j' is of class Hölder α in $\Omega_j \supset \Omega$.

(d) A sector condition on the common endpoints of the arcs.

(e) $|\varphi_j'| \geq C > 0$ in J_j, for each j.

(f) $\varphi_j(\zeta) \neq \varphi_j(z)$ if $\zeta \in J_j$ and $z \in \overline{\Omega}$, $z \neq \zeta$.

Admissible function families, the picture

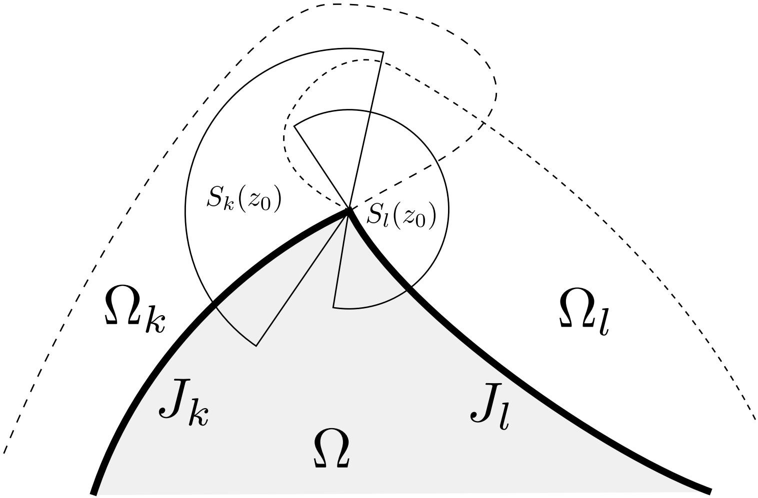

Figure: The geometric properties of an admissible function

Some of our results

Theorem 2.

Let Ω be a simply connected domain and $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible. Suppose that $T \in \mathcal{B}(\mathcal{H})$, and $\sigma(T) \subset \overline{\Omega}$.

(i) *$\Phi = \{\varphi_j\}$ is a strong test collection for $\overline{\Omega}$ (the constant K can depend on T);*

(ii) *if additionally, φ is injective and φ' does not vanish on Ω, then $\Phi = \{\varphi_j\}$ is a strong test collection $($that is, K does not depend on $T)$.*

In particular, if we take Φ to be the collection of conformal Riemann maps of the Ω_js, then this says Φ is a uniform strong test collection.

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In particular, if we take Φ to be the collection of conformal Riemann maps of the Ω_js, then this says Φ is a uniform strong test collection.

Theorem 3.

Let Ω be a domain $($not necessarily connected$)$ and $\varphi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible, with φ injective and φ' not vanishing on Ω. Then φ is a strong test collection.

Havin–Nersessian–Ortega-Cerdà decomposition

- Let $\Omega_1, \ldots, \Omega_n$ be simply connected domains with transversally intersecting $\Omega = \bigcap \Omega_j$. Then every $f \in H^\infty(\Omega)$ has the form

$$f = f_1 + \cdots + f_n, \qquad f_j \in H^\infty(\Omega_j).$$

- In particular, we can write

$$f = g_1 \circ \varphi_1 + \cdots + g_n \circ \varphi_n, \qquad g_j \in H^\infty(\mathbb{D}),$$

where φ_j is the Riemann map for Ω_j (take $g_j = f_j \circ \varphi_j^{-1}$).

A modified HNO-C decomposition

Suppose φ_j maps the boundary section J_j bijectively to an arc in $\mathbb{T}$, but is not necessarily bijective in the interior of Ω_j. If $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is *admissible*, there are bounded operators $F_j : H^\infty(\Omega) \to H^\infty(\mathbb{D})$ such that the operator

$$f \mapsto f - \sum_j F_j(f) \circ \varphi_j$$

is compact with range in $A(\Omega)$ ($H^\infty(\Omega)$ functions which are continuous on $\partial\Omega$). Also, each F_j maps $A(\Omega)$ to $A(\mathbb{D})$.

The algebras $\mathcal{H}_\varphi$ and A_φ

For $\varphi = (\varphi_1, \ldots, \varphi_n)$ as before, define

$$\mathcal{H}_\varphi = \left\{ \sum_{j=1}^{m} \prod_{k=1}^{n} f_{jk} \circ \varphi_j : f_{jk} \in H^\infty(\mathbb{D}) \right\}$$

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What conditions on Ω ensure equality, or more generally, that these are closed subalgebras of finite codimension?

An answer

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If Ω and φ are admissible, then $\mathcal{H}_\varphi$ and A_φ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\Omega)$, respectively.

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Proof: Let $Gf := \sum_j F_j(f) \circ \varphi_j$ on $H^\infty(\Omega)$. So $G - I$ is compact, which implies that $GH^\infty(\Omega) \subset \mathcal{H}_\varphi$ is a closed, finite codimensional subspace in $H^\infty(\Omega)$. Restrict G to $A(\Omega)$ for A_φ. $\square$

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If in addition, φ is injective and φ' does not vanish, we get equality. The proof uses Banach algebra techniques and a classification of the one-codimensional closed unital subalgebras of a unital Banach algebra due to Gorin.

Extensions of functions from analytic curves in $\mathbb{D}^n$

For $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$, the set $\mathcal{V} = \varphi(\Omega)$ is an analytic curve in $\mathbb{D}^n$. We look at $H^\infty(\mathcal{V})$ and $A(\mathcal{V})$. For $f \in H^\infty(\mathcal{V})$, define $\varphi^* f := f \circ \varphi$.

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If Ω and φ are admissible, $\varphi^* H^\infty(\mathcal{V}) = \mathcal{H}_\varphi$ and $\varphi^* A(\mathcal{V}) = A_\varphi$.

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The proof uses the modified HNC-O theorem.

Similarity to normal operators

Stampfli proved in 1969 that if

- $\Gamma \subset \mathbb{C}$ is a smooth curve,
- $T \in \mathcal{B}(\mathcal{H})$ with spectrum $\sigma(T)$ contained in Γ, and
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- Even if Γ is a circle, the condition $\|(T-\lambda)^{-1}\| \leq C\mathrm{dist}(\lambda, \Gamma)^{-1}$, $\lambda \in \mathbb{C} \setminus \Gamma$, where $C > 1$, is not sufficient for T to be *similar* to a normal operator; that is, for some invertible S and normal operator N, to have $T = SNS^{-1}$.

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Nevertheless, the hypothesis in Stampfli's theorem can be successfully weakened.

Similarity to normal operators

We proved the following:

Theorem 6.

Let $\Gamma \subset \mathbb{C}$ be a $C^{1+\alpha}$ Jordan curve, and Ω the domain it bounds. Let $T \in \mathcal{B}(\mathcal{H})$ be an operator with $\sigma(T) \subset \Gamma$. Assume that

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \Gamma)}, \qquad \lambda \in U \setminus \overline{\Omega},$$

for some open set U containing $\partial\Omega$, and

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Other theorems on similarity to normal operators involving resolvent estimates have been proved by van Casteren and Naboko, and we have versions of these as well.

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We also use the following variation on a theorem stated earlier:

Theorem 7.

Let $T \in \mathcal{B}(\mathcal{H})$ and Ω a Jordan domain of class $C^{1+\alpha}$. Assume there is some $R > 0$ such that for every $\lambda \in \partial\Omega$ there is some point $\mu_k(\lambda) \in \mathbb{C} \setminus \overline{\Omega}$ such that $\mathrm{dist}(\mu_k(\lambda), \partial\Omega) = |\mu_k(\lambda) - \lambda| = R$ and $\|(T - \mu_k(\lambda))^{-1}\| \leq R^{-1}$. Then $\overline{\Omega}$ is a complete K-spectral set for some $K > 0$.

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In other words, the conclusion is that there exists a constant $K \geq 1$ such that

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In fact, under the circumstances, we only need to know that Ω is a K-spectral set. However, once we know that T is similar to a normal operator, it follows that Ω is a complete K-spectral set.


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