On complete $K$-spectral sets
(the other title was too long)

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joint with Daniel Estévez and Dmitry Yakubovich

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von Neumann’s Inequality: the basic version

Let $p$ be a complex polynomial and $T \in B(\mathcal{H})$, $\mathcal{H}$ a Hilbert space. If $\|T\| \leq 1$, then $\|p(T)\| \leq \|p\|_\infty$, where $\|p\|_\infty$ is the supremum norm of $p$ over the unit disk $\mathbb{D}$. From the spectral theorem, it is clear von Neumann’s inequality holds if $T$ is a unitary operator. On the other hand, the Sz.-Nagy dilation theorem states that every contraction $T \in B(\mathcal{H})$ has a power dilation $U \in B(K)$ for some Hilbert space $K \supseteq \mathcal{H}$; that is, $T^n = P_{\mathcal{H}}U^n |_{\mathcal{H}}$, $n = 0, 1, 2, \ldots$. Consequently, we obtain the von Neumann inequality as stated.
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Consequently, we obtain the von Neumann inequality as stated.
Spectral sets

Suppose that $\Omega$ is a compact subset of $\mathbb{C}$ and $\sigma(T) \subset \Omega$. Write $R(\Omega)$ for the algebra of functions with poles off of $\Omega$. We say that $\Omega$ is a spectral set for $T$ if for all $r \in R(\Omega)$, $\|r(T)\| \leq \|r\|_{\infty}$, the supremum norm over $\Omega$. Accordingly, von Neumann's inequality may be rephrased as stating that $D$ is a spectral set for any contraction $T$.

Let $K \geq 1$. We will say that $\Omega$ is a $K$-spectral set for $T$ if for all $r \in R(\Omega)$, $\|r(T)\| \leq K \|r\|_{\infty}$. So spectral sets are 1-spectral sets.

We call $\Omega$ a complete spectral (complete $K$-spectral set) for $T$ if these statements hold with $R(\Omega)$ replaced by matrix valued rational functions, so $r \in R(\Omega) \otimes \mathbb{M}_n$ for $n \in \mathbb{N}$.
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The proof is essentially the same as for von Neumann’s inequality. By Runge’s theorem, we can uniformly approximate any function in $R(\overline{D})$ by polynomials, and so it suffices to work with polynomials. Again the spectral theorem allows us to conclude that the result holds whenever $T$ is a unitary operator. The Sz.-Nagy dilation theorem then gives the result for general contractions.
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Hence we have the following:

$T$ is a contraction $\iff \overline{D}$ is a spectral set for $T$ $\iff$
$\overline{D}$ is a complete spectral set for $T$ $\iff T$ dilates to a unitary operator.
We say that an operator $T$ is \emph{polynomially bounded} if it has $\overline{D}$ as an $K$-spectral set for some $K \geq 1$. Similarly it is \emph{completely polynomially bounded} if $\overline{D}$ as a complete $K$-spectral set.
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Suppose $T$ is similar to a contraction; that is, there is a contraction $S$ and an invertible operator $X$ such that $T = X^{-1}SX$. Then one easily sees that for any polynomial $p$,

$$\|p(T)\| \leq \|X^{-1}\| \|X\| \|p(S)\|;$$

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*Halmos’ question:* Is any polynomially bounded operator similar to a contraction?
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In 1997, Pisier, using lacunary sequences and Foguel-type operators,
constructed an example of a polynomially bounded operator which is *not*
similar to a contraction.

There is also a nice later proof due to Davidson and Paulsen, and work
by Badea extending the known counterexamples.
Further results on complete $K$-spectral sets

- *(Douglas-Paulsen, 1986)*: If $\Omega$ is a finitely connected domain with analytic boundary components (given by $|\varphi_k(z)| = 1$), then there is a constant $K$ such that whenever $T \in B(\mathcal{H})$ satisfies $\|\varphi_k(T)\| \leq 1$ for all $k$, then $\Omega$ is a complete $K$-spectral set for $T$.

  (A variation on a theorem of Arveson then implies that $T$ is similar to an operator having a *normal rational $\partial\Omega$-dilation*; that is, $T$ dilates to $N$ *similar* to a normal operator with spectrum on $\partial\Omega$.)
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- *(Badea, Beckermann, Crouzeix, 2009)*: Let $D_j$ be disks in $\hat{\mathbb{C}}$, the extended complex plane, and suppose that these are 1-spectral sets for $T$. Then $\Omega = \bigcap D_j$ is a complete $K$-spectral set for $T$, where $K$ depends only on the number of disks (and not $T$).
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- *(Mascioni, 1994)*: Let $\varphi$ be a finite Blaschke product and suppose that $\overline{D}$ is a complete $K'$-spectral set for $\varphi(T)$. Then $\overline{D}$ is a complete $K$-spectral set for $T$. 


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- (Delyon, Delyon, 1999): Let $\Omega$ be a compact convex set containing the numerical range of an operator $T$. Then $\Omega$ is a complete $K$-spectral set for $T$. 
Realizations and complete $K$-sectral sets

- Let $X \subset \hat{\mathbb{C}}$.
- Let $\Phi = \{ \varphi_j \}$ be a collection of functions analytic on $X$ (the test functions).
- Define $\Omega = \{ z : |\varphi_j(z)| < 1 \}$, $\partial \Omega = \{ z : |\varphi_j(z)| = 1 \}$.
- The admissible kernels are defined as $K_\Phi = \{ k \geq 0 : ((1 - \varphi_j(x)\varphi_j(y))^* k(x, y)) \geq 0 \}$.
- The Agler algebra $H^\infty_d(\mathcal{K}_\Phi)$ of $M_d(\mathbb{C})$-valued analytic functions with norm uniformly bounded as multipliers on all $H^2(k)$, and unit ball $SA_d(\mathcal{K}_\Phi)$ (the Schur-Agler class).
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**Theorem 1 (Realization theorem).**

For $d \in \mathbb{N}$, $f : \Omega \to M_d(\mathbb{C})$, and $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \subset \Omega$, the following are equivalent:

- $f \in SA_d(\mathcal{K}_\Phi)$;
- $\|\varphi_j(T)\| < 1$ for all $k$ implies $\|f(T)\| \leq 1$. 
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- $f \in SA_d(\mathcal{K}_\Phi)$;
- $\|\phi_j(T)\| < 1$ for all $k$ implies $\|f(T)\| \leq 1$.

In other words, any representation of $SA_1(\mathcal{K}_\Phi)$ which is strictly contractive on the test collection is completely contractive.
Our problem

Given

- a test collection $\Phi = \{\varphi_j\}$,
- $\Omega = \{z : |\varphi_j(z)| < 1\}$,
- $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \subset \Omega$,

under what conditions is it the case that

1. If $\|\varphi_j(T)\| \leq 1$ for all $j$ (i.e., $\overline{D}$ is a complete spectral set for $\varphi_j(T)$) or
2. If there exists $K'$ such that for all $j$, $\overline{D}$ is a complete $K'$-spectral set for $\varphi_j(T)$

$\Rightarrow$ there exists $K$ such that $\overline{\Omega}$ is a complete $K$-spectral set for $T$. 
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\[ \Rightarrow \text{there exists } K \text{ such that } \overline{\Omega} \text{ is a complete } K\text{-spectral set for } T. \]

In the latter case, we say that \( \Phi \) is a strong test collection. If \( K \) does not depend on \( T \), \( \Phi \) is called a uniform (strong) test collection. Otherwise it is non-uniform.
Some observations:

- When $\sigma(T) \cap \partial \Omega \neq \emptyset$, things become more difficult!

- If $\Phi = \{\varphi\}$, a uniform test collection is automatically a strong uniform test collection, since then there exists an invertible operator $S$ such that $S\varphi(T)S^{-1} = \varphi(STS^{-1})$ is a contraction.

- $\Phi$ is always a non-uniform strong test collection for $\Omega$ (Rota for $\mathbb{D}$, Herrero & Voiculescu in general).
Some examples of test collections

- The intersection of finitely many disks in \( \widehat{\mathbb{C}} \) (Badea, Beckerman, Crouzeix): *uniform test collection*;
- The *numerical range* of \( T \), \( W(T) = \{ \langle Tx, x \rangle : \|x\| = 1 \} \) is a convex set which is the intersection of (generally infinitely many) closed half planes. The test collection is comprised of the associated linear functionals (Delyon-Delyon and Putinar-Sandberg): *uniform test collection*;
- More generally, \( \rho \)-contractions, \( T^n = \rho P_H U^n |H, n = 1, 2, \ldots \), the intersection of (generally infinitely many) closed disks: *uniform test collection*;
- Nice \( n \)-holed domains (Douglas-Paulsen): *uniform strong test collection*;
- \( \Phi = \{ \varphi \} \), where \( \varphi \) is a finite Blaschke product, \( \Omega = \overline{D} \) (Mascioni): *non-uniform strong test collection*;
- \( \Phi = \{ \varphi \} \), where \( \varphi \) is an infinite Blaschke product with zeros \( \{ \lambda_i \} \) satisfying \( \sum_i (1 - |\lambda_i|^2)^{1/2} < \infty \), \( \Omega = \overline{D} \setminus P \), \( P \) the poles of \( \varphi \) (Stessin): *non-uniform strong test collection*;
Let $\Omega \subset \mathbb{C}$ be a domain whose boundary is a disjoint finite union of piecewise analytic Jordan curves such that the interior angles of the “corners” of $\partial \Omega$ are in $(0, \pi]$. We will say that an analytic function $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{D^n}$ is admissible if $\varphi_j \in \mathbb{A}(\overline{\Omega})$, for $k = 1, \ldots, n$, and there is a collection of closed analytic arcs $\{J_j\}_{k=1}^n$ of $\partial \Omega$ and a constant $\alpha$, $0 < \alpha \leq 1$, such that the following conditions are satisfied:

(a) The arcs $J_j$ cover all $\partial \Omega$.
(b) $|\varphi_j| = 1$ in $J_j$, for $k = 1, \ldots, n$.
(c) For each $j$, ... and $\varphi'_j$ is of class Hölder $\alpha$ in $\Omega_j \supset \Omega$.
(d) A sector condition on the common endpoints of the arcs.
(e) $|\varphi'_j| \geq C > 0$ in $J_j$, for each $j$.
(f) $\varphi_j(\zeta) \neq \varphi_j(z)$ if $\zeta \in J_j$ and $z \in \overline{\Omega}$, $z \neq \zeta$. 

Admissible function families
Admissible function families, the picture

Figure: The geometric properties of an admissible function
Some of our results

**Theorem 2.**

Let \( \Omega \) be a simply connected domain and \( \varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \rightarrow \overline{D}^n \) be admissible. Suppose that \( T \in B(\mathcal{H}) \), and \( \sigma(T) \subset \overline{\Omega} \).

(i) \( \Phi = \{\varphi_j\} \) is a strong test collection for \( \overline{\Omega} \) (the constant \( K \) can depend on \( T \));

(ii) if additionally, \( \varphi \) is injective and \( \varphi' \) does not vanish on \( \Omega \), then \( \Phi = \{\varphi_j\} \) is a strong test collection (that is, \( K \) does not depend on \( T \)).

In particular, if we take \( \Phi \) to be the collection of conformal Riemann maps of the \( \Omega_j \)s, then this says \( \Phi \) is a uniform strong test collection.
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(i) $\Phi = \{\varphi_j\}$ is a strong test collection for $\overline{\Omega}$ (the constant $K$ can depend on $T$);

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In particular, if we take $\Phi$ to be the collection of conformal Riemann maps of the $\Omega_j$'s, then this says $\Phi$ is a uniform strong test collection.

**Theorem 3.**

Let $\Omega$ be a domain (not necessarily connected) and $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{D}^n$ be admissible, with $\varphi$ injective and $\varphi'$ not vanishing on $\Omega$. Then $\varphi$ is a strong test collection.
Let $\Omega_1, \ldots, \Omega_n$ be simply connected domains with transversally intersecting $\Omega = \bigcap \Omega_j$. Then every $f \in H^\infty(\Omega)$ has the form

$$f = f_1 + \cdots + f_n, \quad f_j \in H^\infty(\Omega_j).$$

In particular, we can write

$$f = g_1 \circ \varphi_1 + \cdots + g_n \circ \varphi_n, \quad g_j \in H^\infty(\mathbb{D}),$$

where $\varphi_j$ is the Riemann map for $\Omega_j$ (take $g_j = f_j \circ \varphi_j^{-1}$).
Suppose $\varphi_j$ maps the boundary section $J_j$ bijectively to an arc in $\mathbb{T}$, but is not necessarily bijective in the interior of $\Omega_j$. If $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \mathbb{D}^n$ is admissible, there are bounded operators $F_j : H^\infty(\Omega) \to H^\infty(\mathbb{D})$ such that the operator

$$f \mapsto f - \sum_{j} F_j(f) \circ \varphi_j$$

is compact with range in $A(\Omega)$ ($H^\infty(\Omega)$ functions which are continuous on $\partial\Omega$). Also, each $F_j$ maps $A(\Omega)$ to $A(\mathbb{D})$. 

A modified HNO-C decomposition
The algebras $\mathcal{H}_\varphi$ and $A_\varphi$

For $\varphi = (\varphi_1, \ldots, \varphi_n)$ as before, define

$$\mathcal{H}_\varphi = \left\{ \sum_{j=1}^{m} \prod_{k=1}^{n} f_{jk} \circ \varphi_j : f_{jk} \in H^\infty(\mathbb{D}) \right\}$$

$$A_\varphi = \left\{ \sum_{j=1}^{m} \prod_{k=1}^{n} f_{jk} \circ \varphi_j : f_{jk} \in A(\mathbb{D}) \right\}$$
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What conditions on $\Omega$ ensure equality, or more generally, that these are closed subalgebras of finite codimension?
Theorem 4.

If $\Omega$ and $\varphi$ are admissible, then $H_\varphi$ and $A_\varphi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\Omega)$, respectively.
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Proof: Let $G f := \sum_j F_j(f) \circ \varphi_j$ on $H^\infty(\Omega)$. So $G - I$ is compact, which implies that $GH^\infty(\Omega) \subset H_\varphi$ is a closed, finite codimensional subspace in $H^\infty(\Omega)$. Restrict $G$ to $A(\Omega)$ for $A_\varphi$.  

\qed
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Proof: Let $Gf := \sum_j F_j(f) \circ \varphi_j$ on $H^\infty(\Omega)$. So $G - I$ is compact, which implies that $G H^\infty(\Omega) \subset \mathcal{H}_\varphi$ is a closed, finite codimensional subspace in $H^\infty(\Omega)$. Restrict $G$ to $A(\Omega)$ for $A_\varphi$.

If in addition, $\varphi$ is injective and $\varphi'$ does not vanish, we get equality. The proof uses Banach algebra techniques and a classification of the one-codimensional closed unital subalgebras of a unital Banach algebra due to Gorin.
For $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{D}^n$, the set $\mathcal{V} = \varphi(\Omega)$ is an analytic curve in $D^n$. We look at $H^\infty(\mathcal{V})$ and $A(\mathcal{V})$. For $f \in H^\infty(\mathcal{V})$, define $\varphi^* f := f \circ \varphi$. Theorem 5. There is a $C \geq 1$ such that every $f \in H^\infty(\mathcal{V})$ extends to $F \in SA(D^n)$ with $\|F\| \leq C\|f\|$. The proof uses the modified HNC-O theorem.
For $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{D}^n$, the set $\mathcal{V} = \varphi(\Omega)$ is an analytic curve in $D^n$. We look at $H^\infty(\mathcal{V})$ and $A(\mathcal{V})$. For $f \in H^\infty(\mathcal{V})$, define $\varphi^* f := f \circ \varphi$.

If $\Omega$ and $\varphi$ are admissible, $\varphi^* H^\infty(\mathcal{V}) = \mathcal{H}_\varphi$ and $\varphi^* A(\mathcal{V}) = A_\varphi$. 

Extensions of functions from analytic curves in $\mathbb{D}^n$

For $\varphi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$, the set $\mathcal{V} = \varphi(\Omega)$ is an analytic curve in $\mathbb{D}^n$. We look at $H^\infty(\mathcal{V})$ and $A(\mathcal{V})$. For $f \in H^\infty(\mathcal{V})$, define $\varphi^* f := f \circ \varphi$.

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**Theorem 5.**

There is a $C \geq 1$ such that every $f \in H^\infty(\mathcal{V})$ extends to $F \in SA(\mathbb{D}^n)$ with $\|F\| \leq C\|f\|$. 

The proof uses the modified HNC-O theorem.
For $\varphi = (\varphi_1, \ldots, \varphi_n) : \Omega \rightarrow \overline{D}^n$, the set $\mathcal{V} = \varphi(\Omega)$ is an analytic curve in $\mathbb{D}^n$. We look at $H^\infty(\mathcal{V})$ and $A(\mathcal{V})$. For $f \in H^\infty(\mathcal{V})$, define $\varphi^*f := f \circ \varphi$.

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**Theorem 5.**

*There is a $C \geq 1$ such that every $f \in H^\infty(\mathcal{V})$ extends to $F \in SA(\mathbb{D}^n)$ with $\|F\| \leq C \|f\|$.***

The proof uses the modified HNC-O theorem.
Stampfli proved in 1969 that if

- $\Gamma \subset \mathbb{C}$ is a smooth curve,
- $T \in \mathcal{B}(\mathcal{H})$ with spectrum $\sigma(T)$ contained in $\Gamma$, and
- $U$ a neighborhood of $\Gamma$ such that $\|(T - \lambda)^{-1}\| \leq \text{dist}(\lambda, \Gamma)^{-1}$ for all $\lambda \in U \setminus \Gamma$,

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- If $\Gamma$ is not smooth, a result of this kind need no longer be true.
- Even if $\Gamma$ is a circle, the condition $\|(T - \lambda)^{-1}\| \leq C\text{dist}(\lambda, \Gamma)^{-1}$, $\lambda \in \mathbb{C} \setminus \Gamma$, where $C > 1$, is not sufficient for $T$ to be similar to a normal operator; that is, for some invertible $S$ and normal operator $N$, to have $T = SNS^{-1}$.
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Nevertheless, the hypothesis in Stampfli’s theorem can be successfully weakened.
We proved the following:

**Theorem 6.**

*Let $\Gamma \subset \mathbb{C}$ be a $C^{1+\alpha}$ Jordan curve, and $\Omega$ the domain it bounds. Let $T \in \mathcal{B}(\mathcal{H})$ be an operator with $\sigma(T) \subset \Gamma$. Assume that*

$$
\|(T - \lambda)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, \Gamma)}, \quad \lambda \in U \setminus \overline{\Omega},
$$

*for some open set $U$ containing $\partial \Omega$, and*

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*for some constant $C$. Then $T$ is similar to a normal operator.*
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The same conclusion holds if, vice versa, these estimates hold with with constant 1 inside $\Omega$ and with constant $C$ outside $\Omega$. 
Similarity to normal operators

We proved the following:

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The same conclusion holds if, vice versa, these estimates hold with with constant $1$ inside $\Omega$ and with constant $C$ outside $\Omega$.

Other theorems on similarity to normal operators involving resolvent estimates have been proved by van Casteren and Naboko, and we have versions of these as well.
Idea of the proof

The key technical tool is a generalization of the Riesz-Dunford functional calculus due to Dynkin.

Theorem 7. Let $T \in \mathcal{B}(H)$ and $\Omega$ a Jordan domain of class $C^{1+\alpha}$. Assume there is some $R > 0$ such that for every $\lambda \in \partial \Omega$ there is some point $\mu_k(\lambda) \in \mathbb{C} \setminus \Omega$ such that $\text{dist}(\mu_k(\lambda), \partial \Omega) = |\mu_k(\lambda) - \lambda| = R$ and $\|T - \mu_k(\lambda)\| \leq R - 1$. Then $\Omega$ is a complete $K$-spectral set for some $K > 0$.

In other words, the conclusion is that there exists a constant $K \geq 1$ such that $\|f(T)\| \leq K \|f\|_{H^\infty}(\Omega)$, for every (matrix-valued) rational function $f$ with poles off of $\Omega$ (and hence for every $f$ which is continuous in $\Omega$ and analytic in $\Omega$).

In fact, under the circumstances, we only need to know that $\Omega$ is a $K$-spectral set. However, once we know that $T$ is similar to a normal operator, it follows that $\Omega$ is a complete $K$-spectral set.
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The key technical tool is a generalization of the Riesz-Dunford functional calculus due to Dynkin.

We also use the following variation on a theorem stated earlier:

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Let $T \in \mathcal{B}(\mathcal{H})$ and $\Omega$ a Jordan domain of class $C^{1+\alpha}$. Assume there is some $R > 0$ such that for every $\lambda \in \partial \Omega$ there is some point $\mu_k(\lambda) \in \mathbb{C} \setminus \overline{\Omega}$ such that $\text{dist}(\mu_k(\lambda), \partial \Omega) = |\mu_k(\lambda) - \lambda| = R$ and $\|(T - \mu_k(\lambda))^{-1}\| \leq R^{-1}$. Then $\Omega$ is a complete $K$-spectral set for some $K > 0$. 

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Let $T \in \mathcal{B}(\mathcal{H})$ and $\Omega$ a Jordan domain of class $C^{1+\alpha}$. Assume there is some $R > 0$ such that for every $\lambda \in \partial \Omega$ there is some point $\mu_k(\lambda) \in \mathbb{C} \setminus \overline{\Omega}$ such that $\text{dist}(\mu_k(\lambda), \partial \Omega) = |\mu_k(\lambda) - \lambda| = R$ and $\|(T - \mu_k(\lambda))^{-1}\| \leq R^{-1}$. Then $\Omega$ is a complete $K$-spectral set for some $K > 0$.

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In fact, under the circumstances, we only need to know that $\Omega$ is a $K$-spectral set. However, once we know that $T$ is similar to a normal operator, it follows that $\Omega$ is a complete $K$-spectral set.
Joyeux anniversaire Baruch!