Interpolating sequences in complete Pick spaces

Michael Hartz
joint work with Alexandru Aleman, John M^cCarthy and Stefan Richter

Washington University in St. Louis

Multivariable Operator Theory at the Technion

Interpolating sequences

Let

$$H^\infty = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic and bounded}\}.$$

Definition

A sequence (z_n) in $\mathbb{D}$ is interpolating for H^∞ if for every sequence $(\lambda_n) \in \ell^\infty$, there exists $f \in H^\infty$ with

$$f(z_n) = \lambda_n \quad (n \in \mathbb{N}).$$

Write (z_n) satisfies (IS).

Carleson's interpolation theorem

A sequence (z_n) in $\mathbb{D}$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $f_k \in H^\infty$ with $||f_k||_\infty \leq 1$ and $f_k(z_j) = \varepsilon \delta_{kj}$.

Carleson's interpolation theorem

A sequence (z_n) in $\mathbb{D}$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $f_k \in H^\infty$ with $||f_k||_\infty \leq 1$ and $f_k(z_j) = \varepsilon \delta_{kj}$.

(WS) is weakly separated if there exists $\varepsilon > 0$ such that whenever $j \neq k$, there exists $f_{kj} \in H^\infty$ with $||f_{kj}||_\infty \leq 1$ and $f_{kj}(z_j) = 0$ and $f_{kj}(z_k) = \varepsilon$.

Carleson's interpolation theorem

A sequence (z_n) in $\mathbb{D}$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $f_k \in H^\infty$ with $||f_k||_\infty \le 1$ and $f_k(z_j) = \varepsilon \delta_{kj}$.

(WS) is weakly separated if there exists $\varepsilon > 0$ such that whenever $j \neq k$, there exists $f_{kj} \in H^\infty$ with $||f_{kj}||_\infty \le 1$ and $f_{kj}(z_j) = 0$ and $f_{kj}(z_k) = \varepsilon$.

(C) satisfies the Carleson measure condition if there exists $M > 0$ such that

$$\sum_j (1 - |z_j|^2)|f(z_j)|^2 \le M \int_{\partial\mathbb{D}} |f|^2 \, dm$$

for all $f \in \mathbb{C}[z]$.

Carleson's interpolation theorem

A sequence (z_n) in $\mathbb{D}$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $f_k \in H^\infty$ with $||f_k||_\infty \le 1$ and $f_k(z_j) = \varepsilon \delta_{kj}$.

(WS) is weakly separated if there exists $\varepsilon > 0$ such that whenever $j \neq k$, there exists $f_{kj} \in H^\infty$ with $||f_{kj}||_\infty \le 1$ and $f_{kj}(z_j) = 0$ and $f_{kj}(z_k) = \varepsilon$.

(C) satisfies the Carleson measure condition if there exists $M > 0$ such that

$$\sum_j (1 - |z_j|^2)|f(z_j)|^2 \le M \int_{\partial \mathbb{D}} |f|^2 \, dm$$

for all $f \in \mathbb{C}[z]$.

Theorem (Carleson, 1958)

For a sequence (z_n) in $\mathbb{D}$, (IS) $\Leftrightarrow$ (SS) $\Leftrightarrow$ (WS) + (C).

Why study interpolating sequences?

The maximal ideal space of H^∞

Let

$$\mathfrak{M} = \{\rho : H^\infty \to \mathbb{C} : \rho \text{ is linear, multiplicative}\} \setminus \{0\}$$

and identify $\mathbb{D} \subset \mathfrak{M}$ via point evaluations.

Why study interpolating sequences?

The maximal ideal space of H^∞

Let

$$\mathfrak{M} = \{\rho : H^\infty \to \mathbb{C} : \rho \text{ is linear, multiplicative}\} \setminus \{0\}$$

and identify $\mathbb{D} \subset \mathfrak{M}$ via point evaluations.

Proposition

If (z_n) is an interpolating sequence, then $\overline{\{z_n : n \in \mathbb{N}\}} \subset \mathfrak{M}$ is homeomorphic to $\beta\mathbb{N}$. In particular, $\mathfrak{M}$ is not metrizable and has cardinality $2^{2^{\aleph_0}}$.

Why study interpolating sequences?

The maximal ideal space of H^∞

Let

$$\mathfrak{M} = \{\rho : H^\infty \to \mathbb{C} : \rho \text{ is linear, multiplicative}\} \setminus \{0\}$$

and identify $\mathbb{D} \subset \mathfrak{M}$ via point evaluations.

Proposition

If (z_n) is an interpolating sequence, then $\overline{\{z_n : n \in \mathbb{N}\}} \subset \mathfrak{M}$ is homeomorphic to $\beta\mathbb{N}$. In particular, $\mathfrak{M}$ is not metrizable and has cardinality $2^{2^{\aleph_0}}$.

An analytic disc in $\mathfrak{M}$ is the image of a continuous injection $L : \mathbb{D} \to \mathfrak{M}$ such that $f \circ L$ is analytic for every $f \in H^\infty$.

Theorem (Hoffman, 1967)

A point $m \in \mathfrak{M}$ lies in an analytic disc if and only if it belongs to the closure of an interpolating sequence.

Why study interpolating sequences?

Algebras between H^∞ and L^∞

Let $L^\infty = L^\infty(\partial\mathbb{D})$ and identify $H^\infty \hookrightarrow L^\infty$ via radial boundary values.

Douglas problem (1969)

Characterize closed algebras A with $H^\infty \subset A \subset L^\infty$.

Why study interpolating sequences?

Algebras between H^∞ and L^∞

Let $L^\infty = L^\infty(\partial\mathbb{D})$ and identify $H^\infty \hookrightarrow L^\infty$ via radial boundary values.

Douglas problem (1969)

Characterize closed algebras A with $H^\infty \subset A \subset L^\infty$.

A function $f \in H^\infty$ is inner if $|f| = 1$ a.e. on $\partial\mathbb{D}$.

Theorem (Chang–Marshall, 1976)

If A is a closed algebra with $H^\infty \subset A \subset L^\infty$, then there is a set B of inner functions such that

$$A = \overline{\operatorname{alg}}(H^\infty \cup \{\overline{b} : b \in B\}).$$

Why study interpolating sequences?

Algebras between H^∞ and L^∞

Let $L^\infty = L^\infty(\partial\mathbb{D})$ and identify $H^\infty \hookrightarrow L^\infty$ via radial boundary values.

Douglas problem (1969)

Characterize closed algebras A with $H^\infty \subset A \subset L^\infty$.

A function $f \in H^\infty$ is inner if $|f| = 1$ a.e. on $\partial\mathbb{D}$.

Theorem (Chang–Marshall, 1976)

If A is a closed algebra with $H^\infty \subset A \subset L^\infty$, then there is a set B of inner functions such that

$$A = \overline{\mathrm{alg}}(H^\infty \cup \{\overline{b} : b \in B\}).$$

The functions in B can be chosen to be Blaschke products whose zeros are interpolating sequences.

H^∞ as a multiplier algebra

Let

$$H^2 = \Big\{ f = \sum_{n=0}^\infty a_n z^n \in \mathcal{O}(\mathbb{D}) : ||f||^2 = \sum_{n=0}^\infty |a_n|^2 < \infty \Big\}.$$

This is a reproducing kernel Hilbert space on $\mathbb{D}$: For all $f \in H^2$ and $w \in \mathbb{D}$,

$$f(w) = \langle f, K(\cdot, w) \rangle_{H^2},$$

where

$$K(z,w) = \frac{1}{1 - z\overline{w}}.$$

H^∞ as a multiplier algebra

Let

$$H^2 = \Big\{ f = \sum_{n=0}^\infty a_n z^n \in \mathcal{O}(\mathbb{D}) : ||f||^2 = \sum_{n=0}^\infty |a_n|^2 < \infty \Big\}.$$

This is a reproducing kernel Hilbert space on $\mathbb{D}$: For all $f \in H^2$ and $w \in \mathbb{D}$,

$$f(w) = \langle f, K(\cdot, w) \rangle_{H^2},$$

where

$$K(z,w) = \frac{1}{1 - z\overline{w}}.$$

The multiplier algebra is

$$\operatorname{Mult}(H^2) = \{ \varphi : \mathbb{D} \to \mathbb{C} : \varphi \cdot f \in H^2 \text{ for all } f \in H^2 \},$$

equipped with the multiplier norm $||\varphi||_{\operatorname{Mult}(H^2)} = ||f \mapsto \varphi \cdot f||_{B(H^2)}$.

Fact

$\operatorname{Mult}(H^2) = H^\infty$ with equality of norms.

Using Hilbert function spaces

Shapiro–Shields (1962): Different proof of Carleson's theorem, based on:

Lemma (Shapiro–Shields)

A sequence (z_n) in $\mathbb{D}$ is interpolating for H^∞ if and only if the operator

$$f \mapsto \left(f(z_n)\sqrt{1 - |z_n|^2} \right)$$

maps H^2 onto ℓ^2.

Using Hilbert function spaces

Shapiro–Shields (1962): Different proof of Carleson's theorem, based on:

Lemma (Shapiro–Shields)

A sequence (z_n) in $\mathbb{D}$ is interpolating for H^∞ if and only if the operator

$$f \mapsto \left(f(z_n)\sqrt{1-|z_n|^2} \right)$$

maps H^2 onto ℓ^2.

Bishop, Marshall–Sundberg (1994): Characterize interpolating sequences for the multiplier algebra of the Dirichlet space

$$\mathcal{D} = \{ f \in \mathcal{O}(\mathbb{D}) : f' \in L^2(\mathbb{D}) \}.$$

Using Hilbert function spaces

Shapiro–Shields (1962): Different proof of Carleson's theorem, based on:

Lemma (Shapiro–Shields)

A sequence (z_n) in $\mathbb{D}$ is interpolating for H^∞ if and only if the operator

$$f \mapsto \left(f(z_n) \sqrt{1 - |z_n|^2} \right)$$

maps H^2 onto ℓ^2.

Bishop, Marshall–Sundberg (1994): Characterize interpolating sequences for the multiplier algebra of the Dirichlet space

$$\mathcal{D} = \{ f \in \mathcal{O}(\mathbb{D}) : f' \in L^2(\mathbb{D}) \}.$$

Key property

H^2 and $\mathcal{D}$ are complete Pick spaces.

Nevanlinna–Pick interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \ldots, z_n \in \mathbb{D}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. There exists $f \in H^\infty$ with

$$f(z_i) = \lambda_i \text{ for } 1 \le i \le n \quad \text{and} \quad ||f||_\infty \le 1$$

if and only if the matrix

$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n$$

is positive.

Nevanlinna–Pick interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \dots, z_n \in \mathbb{D}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. There exists $f \in \mathrm{Mult}(H^2)$ with

$$f(z_i) = \lambda_i \text{ for } 1 \le i \le n \quad \text{and} \quad ||f||_{\mathrm{Mult}(H^2)} \le 1$$

if and only if the matrix

$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n$$

is positive.

Nevanlinna–Pick interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \dots, z_n \in \mathbb{D}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. There exists $f \in \mathrm{Mult}(H^2)$ with

$$f(z_i) = \lambda_i \text{ for } 1 \le i \le n \quad \text{and} \quad ||f||_{\mathrm{Mult}(H^2)} \le 1$$

if and only if the matrix

$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}}\right]_{i,j=1}^n = \left[(1 - \lambda_i \overline{\lambda_j}) K(z_i, z_j)\right]_{i,j=1}^n$$

is positive. Here $K(z,w) = (1 - z\overline{w})^{-1}$ is the reproducing kernel of H^2.

Nevanlinna–Pick interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \dots, z_n \in \mathbb{D}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. There exists $f \in \mathrm{Mult}(H^2)$ with

$$f(z_i) = \lambda_i \text{ for } 1 \le i \le n \quad \text{and} \quad ||f||_{\mathrm{Mult}(H^2)} \le 1$$

if and only if the matrix

$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n$$

is positive. Here $K(z,w) = (1 - z\overline{w})^{-1}$ is the reproducing kernel of H^2.

Muhly–Solel (2003): far-reaching generalization to Hardy algebras of W^*-correspondences.

Complete Pick spaces

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel K. Given $z_1, \dots, z_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, does there exist $f \in \mathrm{Mult}(\mathcal{H})$ with

$$f(z_i) = \lambda_i \quad \text{for } 1 \le i \le n \quad \text{and} \quad ||f||_{\mathrm{Mult}(\mathcal{H})} \le 1?$$

Complete Pick spaces

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel K. Given $z_1, \dots, z_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, does there exist $f \in \text{Mult}(\mathcal{H})$ with

$$f(z_i) = \lambda_i \quad \text{for } 1 \le i \le n \quad \text{and} \quad ||f||_{\text{Mult}(\mathcal{H})} \le 1?$$

A necessary condition is that the matrix

$$\left[K(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\right]_{i,j=1}^n$$

is positive.

Definition

$\mathcal{H}$ is called a Pick space if this condition is sufficient. $\mathcal{H}$ is called a complete Pick space if the analogue of this condition for matrix valued functions is sufficient.

Examples

- The Hardy space H^2 is a complete Pick space.

Examples

- The Hardy space H^2 is a complete Pick space.
- The Bergman space $L^2_a = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$ is not a Pick space.

Examples

- The Hardy space H^2 is a complete Pick space.
- The Bergman space $L^2_a = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$ is not a Pick space.
- The Dirichlet space

$$\mathcal{D} = \{f \in \mathcal{O}(\mathbb{D}) : f' \in L^2(\mathbb{D})\},$$

with norm $||f||^2_{\mathcal{D}} = ||f'||^2_{L^2(\mathbb{D})} + ||f||^2_{H^2}$ is a complete Pick space (Agler, 1988).

Examples

- The Hardy space H^2 is a complete Pick space.
- The Bergman space $L^2_a = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$ is not a Pick space.
- The Dirichlet space

  $$\mathcal{D} = \{f \in \mathcal{O}(\mathbb{D}) : f' \in L^2(\mathbb{D})\},$$

  with norm $||f||^2_{\mathcal{D}} = ||f'||^2_{L^2(\mathbb{D})} + ||f||^2_{H^2}$ is a complete Pick space (Agler, 1988).
- The Drury-Arveson space H^2_d is the reproducing kernel Hilbert space on $\mathbb{B}_d$, the open unit ball in $\mathbb{C}^d$, with kernel

  $$K(z, w) = \frac{1}{1 - \langle z, w \rangle}.$$

  This is a complete Pick space.

Interpolating sequences for complete Pick spaces

Let $\mathcal{H}$ be a complete Pick space on X with kernel K. A sequence (z_n) in X

(IS) is an interpolating sequence if for every sequence $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \mathrm{Mult}(\mathcal{H})$ with

$$\varphi(z_n) = \lambda_n \quad (n \in \mathbb{N}).$$

Interpolating sequences for complete Pick spaces

Let $\mathcal{H}$ be a complete Pick space on X with kernel K. A sequence (z_n) in X

(IS) is an interpolating sequence if for every sequence $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \mathrm{Mult}(\mathcal{H})$ with

$$\varphi(z_n) = \lambda_n \quad (n \in \mathbb{N}).$$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $\varphi_k \in \mathrm{Mult}(\mathcal{H})_1$ with $\varphi_k(z_j) = \varepsilon \delta_{kj}$.

Interpolating sequences for complete Pick spaces

Let $\mathcal{H}$ be a complete Pick space on X with kernel K. A sequence (z_n) in X

(IS) is an interpolating sequence if for every sequence $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \mathrm{Mult}(\mathcal{H})$ with

$$\varphi(z_n) = \lambda_n \quad (n \in \mathbb{N}).$$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $\varphi_k \in \mathrm{Mult}(\mathcal{H})_1$ with $\varphi_k(z_j) = \varepsilon \delta_{kj}$.

(WS) is weakly separated if there exists $\varepsilon > 0$ such that whenever $j \neq k$ there exists $\varphi_{kj} \in \mathrm{Mult}(\mathcal{H})_1$ with $\varphi_{kj}(z_j) = 0$ and $f_{kj}(z_k) = \varepsilon$.

Interpolating sequences for complete Pick spaces

Let $\mathcal{H}$ be a complete Pick space on X with kernel K. A sequence (z_n) in X

(IS) is an interpolating sequence if for every sequence $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \text{Mult}(\mathcal{H})$ with

$$\varphi(z_n) = \lambda_n \quad (n \in \mathbb{N}).$$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $\varphi_k \in \text{Mult}(\mathcal{H})_1$ with $\varphi_k(z_j) = \varepsilon \delta_{kj}$.

(WS) is weakly separated if there exists $\varepsilon > 0$ such that whenever $j \neq k$ there exists $\varphi_{kj} \in \text{Mult}(\mathcal{H})_1$ with $\varphi_{kj}(z_j) = 0$ and $f_{kj}(z_k) = \varepsilon$.

(C) satisfies the Carleson measure condition if there exists $M > 0$ such that

$$\sum_j \frac{|f(z_j)|^2}{K(z_j, z_j)} \leq M ||f||^2_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H}.$$

Weak separation

Define a metric on X by

$$d_{\mathcal{H}}(z, w) = \sqrt{1 - \frac{|K(z, w)|^2}{K(z, z)K(w, w)}} \qquad (z, w \in X).$$

Weak separation

Define a metric on X by

$$d_{\mathcal{H}}(z,w) = \sqrt{1 - \frac{|K(z,w)|^2}{K(z,z)K(w,w)}} \qquad (z,w \in X).$$

Example

If $\mathcal{H} = H^2$, then

$$d_{H^2}(z,w) = \left| \frac{z-w}{1-\overline{z}w} \right|$$

is the pseudohyperbolic metric on $\mathbb{D}$.

Weak separation

Define a metric on X by

$$d_{\mathcal{H}}(z,w) = \sqrt{1 - \frac{|K(z,w)|^2}{K(z,z)K(w,w)}} \qquad (z,w \in X).$$

Example

If $\mathcal{H} = H^2$, then

$$d_{H^2}(z,w) = \left| \frac{z-w}{1-\overline{z}w} \right|$$

is the pseudohyperbolic metric on $\mathbb{D}$.

Lemma

If $\mathcal{H}$ is a complete Pick space, then a sequence (z_n) in X satisfies (WS) if and only if there exists $\varepsilon > 0$ such that

$$d_{\mathcal{H}}(z_n, z_m) \geq \varepsilon \quad \text{whenever } n \neq m.$$

Known results about interpolating sequences

Easy facts

In general, (IS) $\Rightarrow$ (SS) and (IS) $\Rightarrow$ (WS) + (C).

Known results about interpolating sequences

Easy facts

In general, (IS) $\Rightarrow$ (SS) and (IS) $\Rightarrow$ (WS) + (C).

Theorem (Bishop, Marshall–Sundberg, 1994)

For the Dirichlet space, (WS) + (C) $\Leftrightarrow$ (IS), but (SS) $\not\Rightarrow$ (IS).

Known results about interpolating sequences

Easy facts

In general, (IS) $\Rightarrow$ (SS) and (IS) $\Rightarrow$ (WS) + (C).

Theorem (Bishop, Marshall–Sundberg, 1994)

For the Dirichlet space, (WS) + (C) $\Leftrightarrow$ (IS), but (SS) $\not\Rightarrow$ (IS).

Theorem (Bøe, 2005)

(WS) + (C) $\Leftrightarrow$ (IS) for every space on the unit ball $\mathbb{B}_d$ with kernel

$$K(z,w) = \frac{1}{(1 - \langle z, w \rangle)^\alpha}, \quad \text{where } \alpha \in (0,1).$$

Known results about interpolating sequences

Easy facts

In general, (IS) $\Rightarrow$ (SS) and (IS) $\Rightarrow$ (WS) + (C).

Theorem (Bishop, Marshall–Sundberg, 1994)

For the Dirichlet space, (WS) + (C) $\Leftrightarrow$ (IS), but (SS) $\not\Rightarrow$ (IS).

Theorem (Bøe, 2005)

(WS) + (C) $\Leftrightarrow$ (IS) for every space on the unit ball $\mathbb{B}_d$ with kernel

$$K(z,w) = \frac{1}{(1 - \langle z, w\rangle)^\alpha}, \qquad \text{where } \alpha \in (0,1).$$

Theorem (Agler–McCarthy, 2002)

(WS) + (C) $\Rightarrow$ (SS) for every complete Pick space.

The main result

Theorem (Aleman, H., M^c^Carthy, Richter)

In every complete Pick space, (WS) + (C) $\Leftrightarrow$ (IS).

The main result

Theorem (Aleman, H., M^{c}Carthy, Richter)

In every complete Pick space, (WS) + (C) $\Leftrightarrow$ (IS).

In this case, there exists a linear operator of operator of interpolation, i.e.

$$\mathrm{Mult}(\mathcal{H}) \to \ell^\infty, \quad \varphi \mapsto (\varphi(z_n)),$$

has a bounded linear right-inverse.

Grammians

Let $\mathcal{H}$ be a complete Pick space on X with kernel K, let (z_n) be a sequence in X. Let $k_i = K(\cdot, z_i)$ and let

$$G[(z_n)] = \Big[\Big\langle \frac{k_i}{||k_i||}, \frac{k_j}{||k_j||} \Big\rangle\Big]_{i,j}$$

be the Grammian.

Proposition

(z_n) satisfies (C) iff $G[(z_n)]$ is bounded.

Grammians

Let $\mathcal{H}$ be a complete Pick space on X with kernel K, let (z_n) be a sequence in X. Let $k_i = K(\cdot, z_i)$ and let

$$G[(z_n)] = \Big[\Big\langle \frac{k_i}{||k_i||}, \frac{k_j}{||k_j||} \Big\rangle\Big]_{i,j}$$

be the Grammian.

Proposition

(z_n) satisfies (C) iff $G[(z_n)]$ is bounded.

Theorem (Marshall–Sundberg, 1994)

(z_n) satisfies (IS) iff $G[(z_n)]$ is bounded and bounded below.

Kadison–Singer

Theorem (Marcus–Spielman–Srivastava, 2013)

Let (v_n) be a sequence of unit vectors in a Hilbert space and let $G = [\langle v_i, v_j \rangle]_{i,j}$ be the Grammian. If G is bounded, then (v_n) is a finite union of sequences whose Grammian is bounded and bounded below.

Kadison–Singer

Theorem (Marcus–Spielman–Srivastava, 2013)

Let (v_n) be a sequence of unit vectors in a Hilbert space and let $G = [\langle v_i, v_j \rangle]_{i,j}$ be the Grammian. If G is bounded, then (v_n) is a finite union of sequences whose Grammian is bounded and bounded below.

Corollary

If (z_n) satisfies (C), then (z_n) is a finite union of sequences that satisfy (IS).

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \text{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \mathrm{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

Let

$$\begin{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & \text{\huge 0} \\ \text{\huge 0} & \begin{bmatrix} \mu_1 & 0 & \cdots \\ 0 & \mu_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \end{bmatrix}$$

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \operatorname{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

Let

$$\begin{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & \Large 0 \\ \Large 0 & \begin{bmatrix} \mu_1 & 0 & \cdots \\ 0 & \mu_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \end{bmatrix}$$

Theorem 1 (Agler–McCarthy, 2002)

$G[(x_n)]$ is bounded below iff there exists a sequence (φ_n) in $\operatorname{Mult}(\mathcal{H})$ such that $[M_{\varphi_1} \; M_{\varphi_2} \; \cdots]$ is bounded and such that $\varphi_k(x_n) = \delta_{nk}$.

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \operatorname{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

Let

$$\begin{bmatrix} \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots \end{bmatrix} & \begin{bmatrix} \psi_1 & \psi_2 & \cdots \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \mu_1 & 0 & \cdots \\ 0 & \mu_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \end{bmatrix}$$

where $\varphi_n(x_n) = 1 = \psi_n(y_n)$

Theorem 1 (Agler–McCarthy, 2002)

$G[(x_n)]$ is bounded below iff there exists a sequence (φ_n) in $\operatorname{Mult}(\mathcal{H})$ such that $[M_{\varphi_1} \; M_{\varphi_2} \; \cdots]$ is bounded and such that $\varphi_k(x_n) = \delta_{nk}$.

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \mathrm{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

Let

$$\begin{bmatrix} \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots \end{bmatrix} & \begin{bmatrix} \psi_1 & \psi_2 & \cdots \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \mu_1 & 0 & \cdots \\ 0 & \mu_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \end{bmatrix}$$

where $\varphi_n(x_n) = 1 = \psi_n(y_n)$

Theorem 2 (Agler–McCarthy, 2002)

(z_n) satisfies (WS) + (C) iff there exists a sequence (φ_n) in $\mathrm{Mult}(\mathcal{H})$ such that $[M_{\varphi_1}\ M_{\varphi_2}\ \cdots]^T$ is bounded and such that $\varphi_k(z_n) = \delta_{nk}$.

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \mathrm{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

Let

$$\varphi = \begin{bmatrix} \begin{bmatrix}\varphi_1 & \varphi_2 & \cdots\end{bmatrix} \begin{bmatrix}\psi_1 & \psi_2 & \cdots\end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \mu_1 & 0 & \cdots \\ 0 & \mu_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \end{bmatrix} \\ \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \end{bmatrix} \end{bmatrix},$$

where $\varphi_n(x_n) = 1 = \psi_n(y_n)$ and $\theta_k(x_n) = \delta_{nk}$ and $\omega_k(x_n) = 0$.

Theorem 2 (Agler–McCarthy, 2002)

(z_n) satisfies (WS) + (C) iff there exists a sequence (φ_n) in $\mathrm{Mult}(\mathcal{H})$ such that $[M_{\varphi_1}\ M_{\varphi_2}\ \cdots]^T$ is bounded and such that $\varphi_k(z_n) = \delta_{nk}$.

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \mathrm{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

Let

$$\varphi = \begin{bmatrix} \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots \end{bmatrix} & \begin{bmatrix} \psi_1 & \psi_2 & \cdots \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \mu_1 & 0 & \cdots \\ 0 & \mu_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \end{bmatrix} \\ \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \end{bmatrix} \end{bmatrix},$$

where $\varphi_n(x_n) = 1 = \psi_n(y_n)$ and $\theta_k(x_n) = \delta_{nk}$ and $\omega_k(x_n) = 0$.

Then $\varphi \in \mathrm{Mult}(\mathcal{H})$ with

$$\varphi(x_n) = \lambda_n \quad \text{and} \quad \varphi(y_n) = \mu_n.$$

Column multipliers and row multipliers

The proof does not require the Marcus–Spielman–Srivastava theorem for every space $\mathcal{H}$ with the following property:

Property (BC) $\Rightarrow$ (BR)

For all sequences (φ_n) in $\mathrm{Mult}(\mathcal{H})$,

$$\begin{bmatrix} M_{\varphi_1} \\ M_{\varphi_2} \\ \vdots \end{bmatrix} \text{ bounded} \quad \Rightarrow \quad \begin{bmatrix} M_{\varphi_1} & M_{\varphi_2} & \cdots \end{bmatrix} \text{ bounded.}$$

This property is satisfied by the Dirichlet space (Trent) and H^2_d ($d < \infty$).

Column multipliers and row multipliers

The proof does not require the Marcus–Spielman–Srivastava theorem for every space $\mathcal{H}$ with the following property:

Property (BC) $\Rightarrow$ (BR)

For all sequences (φ_n) in $\mathrm{Mult}(\mathcal{H})$,

$$\begin{bmatrix} M_{\varphi_1} \\ M_{\varphi_2} \\ \vdots \end{bmatrix} \text{ bounded} \quad \Rightarrow \quad [M_{\varphi_1} \, M_{\varphi_2} \cdots] \text{ bounded.}$$

This property is satisfied by the Dirichlet space (Trent) and H^2_d $(d < \infty)$.

Question

Does every complete Pick space satisfy (BC) $\Rightarrow$ (BR)?

Non-essentially normal multipliers

An operator $T \in \mathcal{B}(\mathcal{H})$ is essentially normal if $TT^* - T^*T$ is compact.

Easy fact

There exists a multiplication operator M_φ on H^2 which is an isometry with infinite dimensional cokernel. In particular, M_φ is not essentially normal.

Non-essentially normal multipliers

An operator $T \in \mathcal{B}(\mathcal{H})$ is essentially normal if $TT^* - T^*T$ is compact.

Easy fact

There exists a multiplication operator M_φ on H^2 which is an isometry with infinite dimensional cokernel. In particular, M_φ is not essentially normal.

Theorem (Lech, 1995)

There exists a multiplication operator on the Dirichlet space $\mathcal{D}$ which is not essentially normal.

Non-essentially normal multipliers

An operator $T \in \mathcal{B}(\mathcal{H})$ is essentially normal if $TT^* - T^*T$ is compact.

Easy fact

There exists a multiplication operator M_φ on H^2 which is an isometry with infinite dimensional cokernel. In particular, M_φ is not essentially normal.

Theorem (Lech, 1995)

There exists a multiplication operator on the Dirichlet space $\mathcal{D}$ which is not essentially normal.

There exist complete Pick spaces (e.g. of continuous functions on $\overline{\mathbb{D}}$) in which every multiplication operator is essentially normal.

Non-essentially normal multipliers: a general result

Proposition (Aleman, H., McCarthy, Richter)

Let $\mathcal{H}$ be a complete Pick space on a connected topological space X with jointly continuous kernel K. If K is unbounded, then there exists a multiplication operator which is not essentially normal.

Non-essentially normal multipliers: a general result

Proposition (Aleman, H., McCarthy, Richter)

Let $\mathcal{H}$ be a complete Pick space on a connected topological space X with jointly continuous kernel K. If K is unbounded, then there exists a multiplication operator which is not essentially normal.

Idea of the proof.

Our characterization of interpolating sequences yields two sequences (x_n) and (y_n) whose union is interpolating such that

$$d_{\mathcal{H}}(x_n, y_n) \leq \frac{1}{2} \quad (n \in \mathbb{N}).$$

Non-essentially normal multipliers: a general result

Proposition (Aleman, H., M^c Carthy, Richter)

Let $\mathcal{H}$ be a complete Pick space on a connected topological space X with jointly continuous kernel K. If K is unbounded, then there exists a multiplication operator which is not essentially normal.

Idea of the proof.

Our characterization of interpolating sequences yields two sequences (x_n) and (y_n) whose union is interpolating such that

$$d_{\mathcal{H}}(x_n, y_n) \leq \frac{1}{2} \quad (n \in \mathbb{N}).$$

Let φ be a multiplier with

$$\varphi(x_n) = 0 \quad \text{and} \quad \varphi(y_n) = 1 \quad (n \in \mathbb{N}).$$

Then M_φ is not essentially normal.

Non-essentially normal multipliers: a general result

Proposition (Aleman, H., McCarthy, Richter)

Let $\mathcal{H}$ be a complete Pick space on a connected topological space X with jointly continuous kernel K. If K is unbounded, then there exists a multiplication operator which is not essentially normal.

Idea of the proof.

Our characterization of interpolating sequences yields two sequences (x_n) and (y_n) whose union is interpolating such that

$$d_{\mathcal{H}}(x_n, y_n) \leq \frac{1}{2} \quad (n \in \mathbb{N}).$$

Let φ be a multiplier with

$$\varphi(x_n) = 0 \quad \text{and} \quad \varphi(y_n) = 1 \quad (n \in \mathbb{N}).$$

Then M_φ is not essentially normal. Pairs of spaces □

Thank you and Happy Birthday to Baruch!

Pairs of spaces

Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be two RKHS on X with kernels K_1, K_2, respectively.

Easy observation

If $\varphi \in \mathrm{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$, then $|\varphi(z)| \le ||\varphi||_M \frac{K_2(z,z)^{1/2}}{K_1(z,z)^{1/2}}$.

Pairs of spaces

Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be two RKHS on X with kernels K_1, K_2, respectively.

Easy observation

If $\varphi \in \text{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$, then $|\varphi(z)| \leq ||\varphi||_M \frac{K_2(z,z)^{1/2}}{K_1(z,z)^{1/2}}$.

Definition

A sequence (z_n) in X is $\text{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$-interpolating if for all $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \text{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$ with $\varphi(z_n) = \frac{K_2(z_n,z_n)^{1/2}}{K_1(z_n,z_n)^{1/2}} \lambda_n$ for all n.

Pairs of spaces

Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be two RKHS on X with kernels K_1, K_2, respectively.

Definition

A sequence (z_n) in X is $\mathrm{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$-interpolating if for all $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \mathrm{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$ with $\varphi(z_n) = \frac{K_2(z_n,z_n)^{1/2}}{K_1(z_n,z_n)^{1/2}} \lambda_n$ for all n.

Theorem (Aleman, H., McCarthy, Richter)

Let $\mathcal{H}_{K_1}, \mathcal{H}_{K_2}$ be two complete Pick spaces on X and let $t \geq 1$. Suppose that $K_2/K_1 \geq 0$. Then a sequence is $\mathrm{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2^t})$ interpolating if and only if it satisfies the $\mathcal{H}_{K_1}$-Carleson condition and is $\mathcal{H}_{K_2}$- weakly separated.

Pairs of spaces

Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be two RKHS on X with kernels K_1, K_2, respectively.

Definition

A sequence (z_n) in X is $\mathrm{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$-interpolating if for all $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \mathrm{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$ with $\varphi(z_n) = \frac{K_2(z_n,z_n)^{1/2}}{K_1(z_n,z_n)^{1/2}} \lambda_n$ for all n.

Theorem (Aleman, H., McCarthy, Richter)

Let $\mathcal{H}_{K_1}, \mathcal{H}_{K_2}$ be two complete Pick spaces on X and let $t \geq 1$. Suppose that $K_2/K_1 \geq 0$. Then a sequence is $\mathrm{Mult}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2^t})$ interpolating if and only if it satisfies the $\mathcal{H}_{K_1}$-Carleson condition and is $\mathcal{H}_{K_2}$- weakly separated.

Corollary

If $\mathcal{H}_K$ is a complete Pick space, then the $\mathrm{Mult}(\mathcal{H}_K)$ and the $\mathrm{Mult}(\mathcal{H}_K, \mathcal{H}_{K^t})$-interpolating sequences agree for $t \geq 1$.

Thank you and Happy Birthday to Baruch!