Operator theory in Drury-Arveson Space

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Introduction

The Drury-Arveson space was introduced by Drury in 1978 and in late 1990s Arveson studied the space systematically.

Stefen Drury

William Arveson

The story of the space goes back to the celebrated von Neumann inequality for contractions.

von Neumann inequality

A linear operator A on a Hilbert space $\mathcal{H}$ is called a contraction if $\|A\| \leq 1$, i.e., if $\|Ax\| \leq \|x\|$ for every vector $x \in \mathcal{H}$.

von Neumann inequality (1951)

If A is a contraction, then

$$\|p(A)\| \leq \sup_{|z|\leq 1} |p(z)|$$

for every polynomial p.

Generalization to operator tuples

Suppose that $(A_1, \ldots, A_d)$ is a commuting tuple of operators and each A_i is a contraction, $i = 1, \ldots, n$. Then one might expect that

$$(*) \qquad \|p(A_1, \ldots, A_d)\| \leq \sup_{|z_1| \leq 1, \ldots, |z_d| \leq 1} |p(z_1, \ldots, z_d)|$$

holds for every d-variable polynomial p.

- This is true in the case $d = 2$ (Ando, 1963) - proved by dilation theorem.
- It is false for $d \geq 3$ in general. (Kaijser and Varopoulos, 1974; Crabb-Davie 1975; Lotto-Steger, 1994; Holbrook, 2001)

Generalization to operator tuples

This raises the following issues:

• What is the right notion of contraction for a commuting tuple of operators?

• What is the right domain on which to consider the problem?

• What is the right norm for the "right-hand side"? (Note that the right-hand side of $(*)$ is $\|p\|_\infty$, the supremum norm of p on the domain in question. Varopoulos' construction shows that for general d, $\|p\|_\infty$ is too small.)

All of these were figured out in late 1970s.

Generalization to operator tuples

- A.Lubin
  - 1976: “Models for commuting contractions.” – developed a dilation theorem for row contractions but did not mention von Neumann inequality
  - 1978: “A subnormal semigroup without normal extension.”
  - 1978: “Research notes on von Neumann inequality.”
- S. Drury
  - 1978: “A generalization of von Neumann’s inequality to the complex ball”

Row contractions

A d-tuple A is said to be commuting if $[A_i, A_j] = 0$ for all i, j.

A commuting tuple $A = (A_1, \ldots, A_d)$ is said to be a row contraction if $A_1 A_1^* + \cdots + A_d A_d^*$ is a contraction.

Equivalently, $(A_1, \ldots, A_d)$ is a row contraction if

$$\|A_1 x_1 + \cdots + A_d x_d\|^2 \leq \|x_1\|^2 + \cdots + \|x_d\|^2$$

for all $x_1, \ldots, x_d \in \mathcal{H}$. Note that this is stronger than requiring that each individual A_i be a contraction, in fact much stronger.

Generalized von Neumann inequality

Drury (1978), Arveson (1998) : If $(A_1, \ldots, A_d)$ is a commuting row contraction, then

$$\|p(A_1, \ldots, A_d)\| \leq \|p\|_{\mathcal{M}} = \|p(M_{z_1}, \ldots, M_{z_d})\|$$

for every $p \in \mathbf{C}[z_1, \ldots, z_d]$.

The right-hand side, $\|p\|_{\mathcal{M}}$, has to be explained on the Drury-Arveson space H^2_d.

Drury-Arveson Space

Basic setting and notation:

$\mathbf{B} = \{z \in \mathbf{C}^d : |z| < 1\}$, the unit ball in $\mathbf{C}^d$.

$S = \{z \in \mathbf{C}^d : |z| = 1\}$, the unit sphere in $\mathbf{C}^d$.

$dv =$ the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$.

$d\sigma =$ the spherical measure on S with the normalization $\sigma(S) = 1$.

Standard multi-index notation:

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}_+^d$ and $z = (z_1, \ldots, z_d) \in \mathbf{B}$, we have

$$\alpha! = \alpha_1! \cdots \alpha_d!,$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_d,$$
$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

BASIC SETTING AND NOTATION:

$A(\mathbf{B})$ = the ball algebra, i.e. the collection of functions which are analytic on $\mathbf{B}$ and continuous on the closed ball $\overline{\mathbf{B}}$.

$L^2_a(\mathbf{B}, dv)$: the Bergman space on $\mathbf{B}$.

$H^2(S)$: the Hardy space, which is a subspace of $L^2(S, d\sigma)$.

H^2_d : the Drury-Arveson space.

Drury-Arveson Space H^2_d

The space H^2_d consists of holomorphic functions on $\mathbf{B}$.

The inner product on H^2_d is defined by

$$\langle f, g \rangle = \sum_{\alpha \in \mathbf{Z}^d_+} \frac{\alpha!}{|\alpha|!} a_\alpha \bar{b}_\alpha$$

for

$$f(z) = \sum_{\alpha \in \mathbf{Z}^d_+} a_\alpha z^\alpha \quad \text{and} \quad g(z) = \sum_{\alpha \in \mathbf{Z}^d_+} b_\alpha z^\alpha.$$

The H^2_d norm is then

$$\|f\| = \sum_{\alpha \in \mathbf{Z}^d_+} \frac{\alpha!}{|\alpha|!} |a_\alpha|^2$$

H_d^2-NORM

The inner product or norm actually arises in several different contexts. For example:

- Apolar bilinear form (19th century invariant theory and differential algebra) / Revitalized by Rota et al.
- Bombieri norm or Bombieri-Weyl norm
- Kostlan norm

H^2_d AS A RKHS

Arveson produced a model theory for row contractions (1998).

- He studied the dilation theory of a commuting contractive d-tuple and came up with the d-tuple of symmetric left creation operators on the symmetric Fock space.
- He observed that the symmetric Fock space is a reproducing-kernel Hilbert space: For $z \in \mathbf{B}$, define

$$K_z(w) = \frac{1}{1 - \langle w, z \rangle}.$$

  K_z is the reproducing kernel for H^2_d.
- He showed that the norm on H^2_d cannot be defined in terms of any measure on $\mathbf{C^d}$, but in a certain sense is maximal among Hilbert space norms on polynomials over the ball.

H_d^2 IN DIFFERENT CONTEXTS

The Drury-Arveson space

- is the symmetric Fock space in d-variables, hence
- a quotient of the full Fock space, hence fits in noncommutative Hardy space theory (Popescu), free semigroup algebra theory (Davidson-Pitts) and
- a baby example of Muhly-Solel setting
- is an example of complete Nevanlinna-Pick interpolation space
- is an example of Besov-Sobolev space
- belongs to a family of reproducing kernel Hilbert modules

For a comprehensive survey on one can check out the survey written by Shalit: *"Operator theory and function theory in Drury-Arveson space"*.

MULTIPLIERS OF H_d^2

One of Arveson's main contributions in his 1998 paper is the introduction of multipliers for H_d^2.

An $f \in H_d^2$ is said to be a multiplier of H_d^2 if

$$fg \in H_d^2 \quad \text{for every } \; g \in H_d^2.$$

We will write $\mathcal{M}$ for the collection of multipliers of H_d^2.

For every $f \in \mathcal{M}$, the multiplication operator M_f is bounded on H_d^2 by the closed graph theorem.

Obviously, we have $\mathbf{C}[z_1, \ldots, z_d] \subset \mathcal{M}$.

MULTIPLIER NORM

For each $f \in \mathcal{M}$, define

$$\|f\|_{\mathcal{M}} = \|M_f\| = \sup\{\|fg\| : g \in H^2_d, \|g\| \leq 1\}.$$

This is called the multiplier norm of f.

Arveson showed that when $d \geq 2$, the collection of multipliers of H^2_d is strictly smaller than H^∞, and the H^∞ norm of a multiplier (even for a polynomial) in general does not dominate the operator norm.

Arveson (1998): There is no L^2 naturally associated with H^2_d. More specifically, the tuple $(M_{z_1}, \cdots, M_{z_d})$ of multiplication by the coordinate functions on H^2_d (d-shift) is not jointly subnormal.

Drury's proof of von Neumann inequality

Recall von Neumann inequality:

$$(**) \qquad \|p(A_1, \ldots, A_d)\| \leq \|p\|_{\mathcal{M}}.$$

It is the multiplier norm of $p \in \mathbf{C}[z_1, \ldots, z_d]$ that appears on the right-hand side.

Drury's proof: In order to prove

$$(**) \qquad \|p(A_1,\dots,A_d)\| \le \|p\|_{\mathcal{M}},$$

it suffices to consider commuting tuples $(A_1,\dots,A_d)$ for which that is an $r \in [0,1)$ such that

$$\langle (A_1A_1^* + \cdots + A_dA_d^*)x, x\rangle \le r^2\|x\|^2, \quad x \in \mathcal{H}.$$

For such a tuple, we can resolve the identity operator 1 in the form

$$\sum_{\alpha \in \mathbf{Z}_+^d} \frac{|\alpha|!}{\alpha!} A^\alpha (1 - A_1A_1^* - \cdots - A_dA_d^*) A^{*\alpha} = 1.$$

This enables us to define an isometry $Z : \mathcal{H} \to H^2_d \otimes \mathcal{H}$:

$$(Zx)(z) = \sum_{\alpha \in \mathbf{Z}^d_+} \frac{|\alpha|!}{\alpha!} (1 - A_1 A_1^* - \cdots - A_d A_d^*)^{1/2} A^{*\alpha} x z^\alpha.$$

It is then straightforward to verify that

$$Zp(A_1^*, \ldots, A_d^*) = (p(M_{z_1}^*, \ldots, M_{z_d}^*) \otimes 1)Z$$

for every $p \in \mathbf{C}[z_1, \ldots, z_d]$. This implies $(**)$.

Essential Norm

The essential norm of an operator A on a Hilbert space $\mathcal{H}$ is defined by

$$\|A\|_{\text{ess}} = \inf\{\|A + K\| : K \text{ is compact on } \mathcal{H}\}.$$

Despite the fact that, when $d \geq 2$, $\|p\|_\infty$ does not dominate the multiplier norm $\|p\|_{\mathcal{M}} = \|M_p\|$, Arveson showed in his 1998 paper that the identity

$$\|M_p\|_{\text{ess}} = \|p\|_\infty$$

holds for every $p \in \mathbf{C}[z_1, \ldots, z_d]$.

ESSENTIAL NORM

Nevertheless:

F. and Xia. (2011): For multipliers $f \in \mathcal{M}$ in general, $\|f\|_\infty$ does not dominate the essential norm $\|M_f\|_{\mathrm{ess}}$ on H^2_d if $d \geq 2$. That is, there is NO $0 < C < \infty$ such that

$$\|M_f\|_{\mathrm{ess}} \leq C\|f\|_\infty \quad \text{for every } f \in \mathcal{M}.$$

This implies that there exist multipliers f of H^2_d for which M_f fails to be essentially hyponormal.

Essential Commutativity

Arveson (1998): For all $i, j \in \{1, \ldots, d\}$, the commutators $[M_{z_i}, M_{z_j}^*]$ are compact on H_d^2. In fact, these operators belong to the Schatten class $\mathcal{C}_p$ for $p > d$.

The corresponding result for the Hardy space is well known. So perhaps one can interpret the above as saying that H_d^2 retains some similarity to the Hardy space. Moreover, we have

F. and Xia. (2011): For every $j \in \{1, \ldots, d\}$ and every $f \in \mathcal{M}$, the commutator $[M_f, M_{z_j}^*]$ on H_d^2 belongs to the Schatten class $\mathcal{C}_p$ for $p > d$. Furthermore, for each $p > d$, there is a $0 < C(p) < \infty$ such that

$$\|[M_f, M_{z_j}^*]\|_p \leq C(p)\|f\|_{\mathcal{M}}$$

for every multiplier $f \in \mathcal{M}$ and every $j \in \{1, \ldots, d\}$, where $\|\cdot\|_p$ is the Schatten p-norm.

Recall that for each $1 \le p < \infty$, the Schatten class $\mathcal{C}_p$ consists of operators A satisfying the condition

$$\|A\|_p = \left\{ \mathrm{tr}((A^*A)^{p/2}) \right\}^{1/p} < \infty.$$

If $A \in \mathcal{C}_p$, then A is compact.
In terms of the s-dumbers $s_1(A)$, $s_2(A)$, ..., $s_j(A)$, ... of A, we have

$$\|A\|_p = \left(\sum_{j=1}^{\infty} \{s_j(A)\}^p \right)^{1/p} .$$

To obtain our Schatten-class result mentioned above, we had to use the fact that even though the tuple $(M_{z_1}, \cdots, M_{z_d})$ is not jointly subnormal, each individual M_{z_j} actually is subnormal on H^2_d.

Carleson Measure

A regular Borel measure $d\mu$ on $\mathbf{B}$ is said to be an Carleson measure for the Drury-Arveson space H^2_d if there is a constant C such that

$$\int |h(z)|^2 d\mu(z) \leq C\|h\|^2$$

for every $h \in H^2_d$. In 2008, Arcozzi, Rochberg and Sawyer gave a characterization for all the Carleson measures for H^2_d.

This characterization is quite complicated, which can be interpreted as a reflection of the structure of H^2_d.

Corona Theorem

Theorem (Costea, Sawyer and Wick 2008)

The corona theorem holds for the multiplier algebra $\mathcal{M}$ of the Drury-Arveson space. That is, for $g_1, \ldots, g_k \in \mathcal{M}$, if there is a $c > 0$ such that

$$|g_1(z)| + \cdots + |g_k(z)| \geq c$$

for every $z \in \mathbf{B}$, then there exist $f_1, \ldots, f_k \in \mathcal{M}$ such that

$$f_1 g_1 + \cdots + f_k g_k = 1.$$

Note: One-function Corona theorem can be proved using more elementary methods (F-Xia 2013, Richter-Sunkes 2016)

More recent developments

- Dilation theorem – Arveson, Müller-Vasilescu
- Nevanlinna-Pick type interpolation problems for contractive multipliers (Schur class multipliers) related to H^2_d have been intensively studied over the past 20 years .
- Commutant lifting theorem (Ball-Trent-Vinnikov, Davidson-Le)
- Duality, convexity and peak interpolation in H^2_d – Clouâtre and Davidson 2016 have established analogues of classic results of the ball algebra to the multiplier algebra.
- Henkin measure for H^2_d – Clouâtre and Davidson 2016, Hartz 2017
- Quasi-extremal multipliers of H^2_d/ Aleksandrov-Clark theory - Jury 2014/Jury-Martin 2017
- Connection between H^2_d and Hilbert spaces of Dirichlet series (McCarthy-Shalit)

Discussions on some problems

The Reciprocal Problem

This is a really elementary problem, but one to which we do not have a general answer. This illustrates how little we know about the Drury-Arveson space.

Question: Let $f \in H^2_d$. Suppose that there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$. Does it follow that $1/f \in H^2_d$?

The answer is "yes" for $d = 2, 3$ by an argument due to Richter and Sundberg. The answer is "yes" with extra Bloch space condition was proved by Richter and Sunkes.

But the problem is open for $d \geq 4$.

Multiplier Characterization

Let $k \in \mathbf{N}$ be such that $2k \geq d$. Then given any $f \in H^2_d$, one can define the measure $d\mu_f$ on $\mathbf{B}$ by the formula

$$(***) \qquad d\mu_f(z) = |(R^k f)(z)|^2 (1 - |z|^2)^{2k-d} dv(z),$$

where $R = z_1\partial_1 + \cdots + z_d\partial_d$, the radial derivative, and dv is the normalized volume measure on $\mathbf{B}$.

In 2000, Ortega and Fàbrega showed that $f \in \mathcal{M}$ if and only if $d\mu_f$ is a Carleson measure for H^2_d.

Recall that in 2008, Arcozzi, Rochberg and Sawyer characterized all Carleson measures for H^2_d.

So the combination of the result of Arcozzi, Rochberg and Sawyer and the result of Ortega and Fàbrega is *a* characterization of the membership $f \in \mathcal{M}$.

We have been trying to look for simpler or more direct, characterization of the membership $f \in \mathcal{M}$ but there is no success so far. Actually:

F. and Xia. (2015): There exists an $f \in H_d^2$ satisfying the conditions $f \notin \mathcal{M}$ and

$$\sup_{|z|<1} \|fk_z\| < \infty.$$

So it shows that the condition $\sup_{|z|<1} \|fk_z\| < \infty$ we desired does not imply the membership $f \in \mathcal{M}$.

Our construction also yields the following negative result:

There does not exist any constant $0 < C < \infty$ *such that the inequality*

$$\|M_f^* M_f - M_f M_f^*\| \leq C \sup_{|z|<1} \|(f - \langle f k_z, k_z \rangle) k_z\|^2$$

holds for every $f \in \mathcal{M}$, *where* M_f *is the operator of multiplication by* f *on* H_d^2.

The quantity $\|(f - \langle f k_z, k_z \rangle) k_z\|$ is the "mean oscillation" of $f \in \mathcal{M}$ with respect to the normalized reproducing kernel of the Drury-Arveson space. And, for those who are familiar with Hankel operators, the commutator $M_f^* M_f - M_f M_f^*$ is the Drury-Arveson space analogue of

$$H_{\bar{f}}^* H_{\bar{f}}.$$

Also we know that the norm inequality

$$(\#) \qquad \|H_\varphi\| \le C\|\varphi\|_{\mathrm{BMO}},$$

holds in the setting of either the Hardy space of the unit sphere or the Bergman space of the unit ball. In contrast, The above negative result tells us that the Drury-Arveson space analogue of $(\#)$ fails. This sets the Drury-Arveson space even farther apart from the Hardy space and the Bergman space.

Invariant subspaces

Arveson (2000): Let N be a closed linear subspace of H_d^2 that is invariant under the multiplication by the polynomials. If $N \neq \{0\}$, then d contains a nonzero multiplier of H_d^2.

An immediate consequence: for invariant subspaces N_1, N_2 of H_d^2, if $N_1 \neq \{0\}$ and $N_2 \neq \{0\}$, then $N_1 \cap N_2 \neq \{0\}$.

Question: For $d \geq 2$, do we have the analogue of this intersection property for invariant subspaces of the Hardy space or Bergman space?

This is a rare situation where we actually know more about the Drury-Arveson space than we know about the more familiar Hardy space or Bergman space.

Reproducing Kernel HS

difficult

easy

Drury Arveson

$K_z(w) = \frac{1}{1-\langle w,z\rangle}$

many operator-theoretic problems

some problems

Hardy

$K_z(w) = \frac{1}{(1-\langle w,z\rangle)^d}$

Bergman

$K_z(w) = \frac{1}{(1-\langle w,z\rangle)^{d+1}}$

easy

difficult

A family of RKHS $\mathcal{H}^{(t)}$

For each real number $-d \leq t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$\frac{1}{(1 - \langle z, w \rangle)^{n+1+t}}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \ldots, z_d]$ with respect to the norm $\| \cdot \|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|} (n + t + j)}$$

if $\alpha \in \mathbf{Z}_+^d \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$.

One can think of the parameter t as the "weight" of the space, although t can be negative.

We have

$$\mathcal{H}^{(0)} = L_a^2(\mathbf{B}, dv), \qquad \text{the Bergman space,}$$

$$\mathcal{H}^{(-1)} = H^2(S), \qquad \text{the Hardy space,}$$

$$\mathcal{H}^{(-d)} = H_d^2, \qquad \text{the Drury-Arveson space.}$$

One can think of the Bergman space $\mathcal{H}^{(0)}$ as a benchmark, against which the other spaces in the family should be compared.

REPRODUCING-KERNEL HILBERT MODULES

These spaces are all Hilbert modules over $\mathbf{C}[z_1, \ldots, z_d]$ under the identification of each $f \in \mathbf{C}[z_1, \ldots, z_d]$ with the multiplication operator M_f.

$L_a^2(\mathbf{B}, dv)$ and $H^2(S)$ are actually Hilbert modules over the ball algebra $A(\mathbf{B})$, but H_d^2 is not.

A submodule is a closed linear subspace $\mathcal{M}$ that is invariant under $M_{z_1}, \ldots, M_{z_d}$. We denote M_{z_i} as Z_i.

Survey on Hilbert module approach - *"An introduction to Hilbert module approach to multivariable operator theory"* by Sarkar

Defect operators associated with submodules

RESTRICTED MODULE OPERATORS

Here is another example of problem where Drury-Arveson case is easier than Hardy/Bergman cases.

Given a submodule $\mathcal{M}$ of a reproducing kernel Hilbert module $\mathcal{H}^t$, for example, the Hardy module, we have the restricted module operators

$$Z_{\mathcal{M},i} = Z_i|\mathcal{M}, \quad i = 1, \ldots, n.$$

A natural question about the submodules is the Schatten class membership, or the lack thereof, of the commutators $[Z^*_{\mathcal{M},i}, Z_{\mathcal{M},j}]$.

Defect operator

It is well known that if $p > d$, then $[Z_i^*, Z_j] \in \mathcal{C}_p$ for all $i, j \in \{1, ..., d\}$.

It is also well known that for each $i \in \{1, ..., d\}$, $[Z_i^*, Z_i] \notin \mathcal{C}_d$.

This leads to the natural question, what happens if we consider $Z_{\mathcal{M},1}, ..., Z_{\mathcal{M},d}$ instead of $Z_1, ..., Z_d$?

Given a submodule $\mathcal{M}$, let us denote

$$D_{\mathcal{M}} = \sum_{i=1}^{d} [Z_{\mathcal{M},i}^*, Z_{\mathcal{M},i}].$$

Drury-Arveson case

Theorem (F.-Xia 2011)

Let $\mathcal{M}$ be any submodule of the Drury-Arveson module H^2_d. If $\mathcal{M} \neq \{0\}$, then there is a positive number $\epsilon = \epsilon(\mathcal{M}) > 0$ such that

$$s_1(D_{\mathcal{M}}) + \ldots + s_k(D_{\mathcal{M}}) \geq \epsilon k^{(d-1)/d}$$

for every $k \in \mathbf{N}$. Consequently, $D_{\mathcal{M}}$ does not belong to the Schatten class $\mathcal{C}_d$ whenever $\mathcal{M} \neq \{0\}$.

HARDY CASE

Theorem (F.-Xia 2011)

Let $\mathcal{M}$ be any submodule of the Hardy module $H^2(S)$. If $\mathcal{M} \neq \{0\}$, then there is a positive number $\epsilon = \epsilon(\mathcal{M}) > 0$ such that

$$s_1(D_{\mathcal{M}}) + \ldots + s_k(D_{\mathcal{M}}) \geq \epsilon k^{(d-1)/d}$$

for every $k \in \mathbf{N}$. Consequently, $D_{\mathcal{M}}$ does not belong to the Schatten class $\mathcal{C}_d$ whenever $\mathcal{M} \neq \{0\}$.

Remark. In the above theorems, the conclusion $D_{\mathcal{M}} \notin \mathcal{C}_d$ is an immediate consequence of the inequality

$$s_1(D_{\mathcal{M}}) + \ldots + s_k(D_{\mathcal{M}}) \geq \epsilon k^{(d-1)/d}, \quad k \in \mathbf{N}$$

This is because, if $1 < p < \infty$ and if $\{a_k\} \in \ell_+^p$, then

$$\lim_{k\to\infty} \frac{1}{k^{(p-1)/p}} \sum_{j=1}^{k} a_j = 0.$$

Motivation for the Investigation

The first motivation is related to what is now commonly referred to as the Arveson conjecture. Simply stated, it is this:

Question. For a submodule $\mathcal{M}$ of H^2_d, do the commutators $[Z^*_{\mathcal{M},i}, Z_{\mathcal{M},j}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > d$?

Later, Douglas proposed the analogous problem for the Bergman space.

From there it takes no imagination for one to think about the case of the Hardy space $H^2(S)$, since all of these are reproducing-kernel Hilbert spaces.

Motivation for the Investigation

In all these versions of the problem, one conspicuous feature is the lower limit $p > d$ that one sets for the Schatten class.

One might say that this lower limit is dictated by known examples. For instance, it is well known that $[Z_i^*, Z_i] \notin \mathcal{C}_d$ on $H^2(S)$ and H_d^2, and the same is also true on the Bergman space of the ball $\mathbf{B}$.

In other words, examples show that the lower limit $p > d$ in these conjectures is necessary for SOME submodules.

The first motivation for this investigation was to find out whether the lower limit $p > d$ is necessary for EVERY submodule $\mathcal{M} \neq \{0\}$.

Motivation for the Investigation

The second motivation is related to extensions of the C^*-algebra $C(S)$ by the compact operators, stemming from a paper of Douglas and Voiculescu in 1981:

Let $(A_1, ..., A_d)$ be an essentially commuting tuple of bounded operators on a separable Hilbert space $\mathcal{H}$. Suppose that $(A_1, ..., A_d)$ generates an exact sequence:

$$\{0\} \to \mathcal{K} \to \mathcal{T} \xrightarrow{\tau} C(S) \to \{0\},$$

where $\mathcal{K}$ is the collection of compact operators, $\mathcal{T}$ is the C^*-algebra generated by A_1, ..., A_d and $\mathcal{K}$, and the homomorphism $\tau : \mathcal{T} \to C(S)$ has the property

$$\tau(A_i) = z_i, \qquad i = 1, \ldots, d.$$

Such an exact sequence represents an element $[\tau]$ in $\operatorname{Ext}(S)$. The class $[\tau]$ can be determined in the following way. There exists a $2^d \times 2^d$ matrix α whose entries are polynomials in $2d$ variables such that if we set

$$A = \alpha(A_1, A_1^*, ..., A_d, A_d^*), \tag{0.1}$$

then A is Fredholm and, under the identification $\operatorname{Ext}(S) \cong \mathbf{Z}$, we have

$$[\tau] = \operatorname{index}(A).$$

Douglas and Voiculescu proved the following index formula, which is generally regarded as a precursor to what is now called non-commutative geometry.

Motivation for the investigation

Theorem (Douglas and Voiculescu)

Suppose that the operators A_1, ..., A_d satisfy the conditions

$$[A_i, A_j] \in \mathcal{C}_d, \quad [A_i^*, A_j] \in \mathcal{C}_d \quad \textit{for all} \;\; 1 \leq i, j \leq d \tag{0.2}$$

and

$$1 - \sum_{i=1}^{d} A_i^* A_i \in \mathcal{C}_d. \tag{0.3}$$

Then for the operator A given by (0.1) *we have*

$$\mathrm{index}(A) = \mathrm{tr}[A_1, A_1^*, ..., A_d, A_d^*], \tag{0.4}$$

where $[A_1, A_1^*, ..., A_d, A_d^*]$ is the antisymmetric sum of A_1, A_1^*, ..., A_d, A_d^*.

Motivation for the investigation

When this index theorem was published, it was not known whether one can have $\mathrm{index}(A) \neq 0$ for a tuple $(A_1, ..., A_d)$ satisfying conditions (0.2) and (0.3).

Later, however, Gong showed that for any $m \in \mathbf{Z}$, there exists a tuple $(A_1, \ldots, A_d)$ satisfying (0.2) and (0.3) with

$$\mathrm{index}(A) = m.$$

Indeed Gong showed that one can even replace Schatten class $\mathcal{C}_d$ with $\mathcal{C}_p$ for any $p > d - (1/2)$.

Motivation for the investigation

But Gong's paper does not tell us whether index formula (0.4) can ever be applied to canonical operator tuples such as

$$(A_1, ..., A_d) = (Z_{\mathcal{M},1}, ..., Z_{\mathcal{M},d}). \tag{0.5}$$

Thus one of the motivating questions for us was whether index formula (0.4) can ever be applied in the case of (0.5) .

Although it is generally believed that the answer is always negative for submodules $\mathcal{M} \neq \{0\}$ of the Hardy, Drury-Arveson and Bergman modules, we did not see earlier results.

About the proofs

- Drury Arveson case
  - Freeness of H^2_d as a Hilbert module over $\mathbf{C}[z_1, ..., z_d]$
  - If $\mathcal{M}$ is a submodule of H^2_d and $\mathcal{M} \neq \{0\}$, then $\mathcal{M}$ contains a non-trivial multiplier of H^2_d
- Hardy case (also solved but much more difficult)
  - A multiplier for the Hardy space $H^2(\mathbf{S})$ is a function in $H^\infty(\mathbf{S})$.
  - If $\mathcal{M}$ is a submodule of the Hardy module, it is not known whether $\mathcal{M} \cap H^\infty(\mathbf{S})$ contains anything other than 0.
  - We were lucky to find a way to get around this unboundedness.

Easy Proof for Drury-Arveson case

Let $\{e_\alpha : \alpha \in \mathbf{Z}_+^d\}$ be the standard orthonormal basis for the Drury-Arveson space H_d^2. That is, for each $\alpha \in \mathbf{Z}_+^d$, $e_\alpha(z) = \left(\frac{|\alpha|!}{\alpha!}\right)^{1/2} z^\alpha$.

On H_d^2 we also define the operator $D = \sum_{i=1}^d [Z_i^*, Z_i]$.

The following identity follows from Arveson's 1998 paper:

Lemma. *For each* $f = \sum_{\alpha \in \mathbf{Z}_+^d} b_\alpha e_\alpha \in H_d^2$ *we have*

$$\langle Df, f\rangle = |b_0|^2 + (d-1) \sum_{\alpha \in \mathbf{Z}_+^d} \frac{|b_\alpha|^2}{|\alpha| + 1}.$$

Easy Proof for Drury-Arveson case

Proof. Let $\mathcal{M}$ be a submodule of H_d^2 and suppose $\mathcal{M} \neq \{0\}$. Then by a result of Arveson, there is a $\varphi \in \mathcal{M}$, $\varphi \neq 0$, which is a multiplier of H_d^2. Then M_φ is bounded on H_d^2.

For each $\ell \in \mathbf{N}$, let $A_\ell = \{(\alpha_1, ..., \alpha_d) : \ell < \alpha_i \leq 2\ell, 1 \leq i \leq d\}$. We then define

$$F_\ell = \sum_{\alpha \in A_\ell} (\varphi e_\alpha) \otimes (\varphi e_\alpha),$$

Since $F_\ell = M_\varphi Q_\ell M_\varphi^*$, where $Q_\ell = \sum_{\alpha \in A_\ell} e_\alpha \otimes e_\alpha$, which is an orthogonal projection, we have

$$\|F_\ell\| \leq \|M_\varphi\|^2$$

Suppose that

$$\varphi = \sum_{\beta \in \mathbf{Z}_+^d} c_\beta e_\beta.$$

Easy Proof for Drury-Arveson case

Applying the above lemma, we have

$$\begin{aligned}\mathrm{tr}(D_{\mathcal{M}}F_\ell) &= \sum_{\alpha\in A_\ell}\langle D_{\mathcal{M}}\varphi e_\alpha, \varphi e_\alpha\rangle \geq \sum_{\alpha\in A_\ell}\langle D\varphi e_\alpha, \varphi e_\alpha\rangle\\ &= (d-1)\sum_{\alpha\in A_\ell}\sum_{\beta\in\mathbf{Z}_+^d}\frac{|c_\beta|^2 u^2(\alpha,\beta)}{|\alpha+\beta|+1},\end{aligned}$$

where

$$u^2(\alpha,\beta) = \frac{|\beta|!}{\beta!}\cdot\frac{|\alpha|!}{\alpha!}\cdot\frac{(\alpha+\beta)!}{|\alpha+\beta|!}.$$

Since $\varphi \neq 0$, there is a $\beta_0 \in \mathbf{Z}_+^d$ such that $c_{\beta_0} \neq 0$. We have

$$\mathrm{tr}(D_{\mathcal{M}}F_\ell) \geq (d-1)|c_{\beta_0}|^2\sum_{\alpha\in A_\ell}\frac{u^2(\alpha,\beta_0)}{|\alpha+\beta_0|+1}.$$

Easy Proof for Drury-Arveson case

Now suppose $\ell > |\beta_0|$. Then for $\alpha \in A_\ell$, we have $|\alpha + \beta_0| + 1 \leq 3d\ell$ and $u^2(\alpha, \beta_0) \geq (|\beta_0|!/\beta_0!) \cdot (1/3n)^{|\beta_0|}$. Hence

$$\mathrm{tr}(D_{\mathcal{M}} F_\ell) \geq \frac{(d-1)|c_{\beta_0}|^2 |\beta_0|!}{(3n)^{|\beta_0|}\beta_0!} \cdot \frac{1}{3n\ell} \cdot \mathrm{card}(A_\ell) = \delta_1 \ell^{d-1}$$

for each $\ell > |\beta_0|$. On the other hand,

$$\mathrm{tr}(D_{\mathcal{M}} F_\ell) \leq \|D_{\mathcal{M}} F_\ell\|_1 = \sum_{j=1}^{\mathrm{rank}(F_\ell)} s_j(D_{\mathcal{M}} F_\ell) \leq \sum_{j=1}^{\ell^d} s_j(D_{\mathcal{M}}) \|F_\ell\|.$$

Since $\|F_\ell\| \leq \|M_\varphi\|^2$, the above yields

$$\sum_{j=1}^{\ell^d} s_j(D_{\mathcal{M}}) \geq \frac{\delta_1}{\|M_\varphi\|^2} \ell^{d-1}$$

for each $\ell > |\beta_0|$. The desired result follows from this. □

Basic idea for Hardy case

Denote

$$D = \sum_{i=1}^{d} [Z_i^*, Z_i].$$

Given a submodule $\mathcal{M}$ of $H^2(S)$, let $P_{\mathcal{M}} : H^2(S) \to \mathcal{M}$ be the orthogonal projection. For $h \in \mathcal{M}$, $Z_{\mathcal{M},i}^* h = P_{\mathcal{M}} Z_i^* h$ and $Z_{\mathcal{M},i} h = Z_i h$, which leads to

$$\begin{aligned} \langle [Z_{\mathcal{M},i}^*, Z_{\mathcal{M},i}] h, h \rangle &= \|Z_{\mathcal{M},i} h\|^2 - \|Z_{\mathcal{M},i}^* h\|^2 \\ &\geq \|Z_i h\|^2 - \|Z_i^* h\|^2 = \langle [Z_i^*, Z_i] h, h \rangle. \end{aligned}$$

Therefore

$$\langle D_{\mathcal{M}} h, h \rangle \geq \langle Dh, h \rangle \quad \text{for every } \ h \in \mathcal{M}. \tag{0.6}$$

Basic idea for Hardy case

Our proof for the Hardy case is based on the realization that, with enough work and further exploitation of the invariance of $\mathcal{M}$ under the multiplication by functions in $A(\mathbf{B})$, Theorem 1.1 can be deduced from (0.6).

To prove the Hardy case, for each $k \in \mathbf{N}$ we need to construct an operator F_k on $\mathcal{M}$ with

$$\mathrm{rank}(F_k) \approx k,$$

$$\mathrm{tr}(D_{\mathcal{M}} F_k) \approx k^{(d-1)/d}$$

and

$$\|F_k\| \leq C, \tag{0.7}$$

where C is independent of k. Of the three requirements, (0.7) turns out to be the biggest obstacle.

Bergman case

Question. Does the analogue of two theorems above hold true for submodules of the Bergman module?

Question. Is there a "unified" method to treat this type of problems for submodules of more general reproducing kernel Hilbert modules?

Essential Normality of polynomial-generated submodules

Arveson's Conjecture

Arveson's Conjecture (circa 2000):

Every graded submodule $\mathcal{M}$ of H^2_d is p-essentially normal for $p > d$, i.e. For a submodule $\mathcal{M}$ of H^2_d, do the commutators $[Z^*_{\mathcal{M},i}, Z_{\mathcal{M},j}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > d$?

Partial list of people who have worked on this and related problems:
Arveson, Douglas, Engliš, Eschmeier, Guo, Kennedy, Shalit, Tang, K.Wang., P.Wang, Y.Wang, Yu

Arveson's Conjecture

Graded: the submodule has an orthogonal decomposition in terms of degree. But the problem becomes much more interesting and challenging for submodules that have NO such orthogonal decomposition.

Arveson first verified his conjecture in the case where the submodule is generated by a finite set of monomials (2005).

Later in 2006, Douglas proposed a similar but more refined conjecture for submodules of the Bergman module $\mathcal{H}^{(0)} = L^2(\mathbf{B}, dv)$.

Guo and Wang (2008): In the case $d = 2, 3$, every graded submodule of H_d^2 is p-essentially normal for $p > d$. For arbitrary d, every submodule of H_d^2 generated by a single homogeneous polynomial is p-essentially normal for $p > d$.

In a quite unexpected development early 2011, Douglas and K.Wang proved

Theorem.(Douglas and Wang)

If $[q]$ is the submodule of the Bergman module $L_a^2(\mathbf{B}, dv)$ generated by any $q \in \mathbf{C}[z_1, \ldots, z_d]$, then $[q]$ is p-essentially normal for $p > d$.

This is an unconditional result in the sense that no assumption is made about the polynomial $q \in \mathbf{C}[z_1, \ldots, z_d]$. This signals the beginning of a new phase of investigations where one moves away from *degree-related* assumptions such as homogeneity.

GEOMETRIC VERSION

Some of recent developments in this line of investigations

"Geometric Arveson-Douglas conjecture" by Engliš and Eschmeier.

"An analytic Grothendieck Riemann Roch theorem" by Douglas, Tang and Yu.

"Geometirc Arveson-Douglas Conjecture - Decomposition of Varieties" by Douglas, Y.Wang

Some related Results

Inspired by the Douglas-K.Wang paper, we decided to take a look at the essential normality of principal submodules of the Hardy module and the Drury-Arveson module.

The key realization is to treat the three modules in a unified way as mentioned before (consider $\mathcal{H}^{(t)}$).

That is, these three spaces are all members of a family of reproducing-kernel Hilbert spaces (modules) of analytic functions on $\mathbf{B}$ parametrized by a real-valued parameter (weight) $-d \leq t < \infty$.

UNCONDITIONAL RESULT

Theorem (F-Xia preprint)

Let q be an arbitrary polynomial in $\mathbf{C}[z_1, \ldots, z_d]$. Then for each real number $-3 < t < \infty$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is p-essentially normal for every $p > d$.

Corollary. ($t = -1$) The submodule of the Hardy module $H^2(S)$ generated by any $q \in \mathbf{C}[z_1, \ldots, z_d]$ is p-essentially normal for every $p > d$.

If we apply this to the case $t = -2$, we obtain the first non-trivial Drury-Arveson space case:

Corollary. ($d = 2$.) The submodule of H^2_2 generated by any $q \in \mathbf{C}[z_1, z_2]$ is p-essentially normal for every $p > 2$.

$t = -3$ CASE

Obviously, we would like to show that for every $q \in \mathbf{C}[z_1, \ldots, z_d]$, the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is essentially normal. This goal we have NOT achieved yet.

Instead, we are able to show that there is a substantial subclass $\mathcal{G}_d$ of $\mathbf{C}[z_1, \ldots, z_d]$ such that for every $q \in \mathcal{G}_d$, the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is essentially normal. An important feature of

the class $\mathcal{G}_d$ is that its membership is stable under small perturbation, in a sense to be made clear later.

$t = -3$ CASE

To tackle the case $t = -3$, we need to consider the zero locus of q.

Given any $q \in \mathbf{C}[z_1, \ldots, z_d]$, we write

$$\mathcal{Z}(q) = \{z \in \mathbf{C}^d : q(z) = 0\}.$$

Write $\partial_1, \ldots, \partial_d$ for the differentiations with respect to the complex variables $z_1, \ldots, z_d$.

Recall that the d-variable radial derivative is given by the formula

$$R = z_1\partial_1 + \cdots + z_d\partial_d.$$

$t = -3$ CASE

Definition. Let $\mathcal{G}_d$ be the collection of polynomials $q \in \mathbf{C}[z_1, \ldots, z_d]$ satisfying the following two conditions:
(a) The radial derivative Rq does not vanish on the set $\mathcal{Z}(q) \cap S$.
(b) The zero locus $\mathcal{Z}(q)$ intersects the unit sphere S transversely.

Conditions (a), (b) above are inspired by Assumption 1.1 in the Douglas-Tang-Yu paper mentioned earlier.

Note that condition (a) implies that the analytic gradient $\partial q = (\partial_1 q, \ldots, \partial_d q)$ does not vanish on the set $\mathcal{Z}(q) \cap S$, which ensures that (b) makes sense. At every point in S, the (real) co-dimension of the tangent space to S is 1. Thus condition (b) is simply equivalent to the condition that if $\xi \in \mathcal{Z}(q) \cap S$, then the tangent space to $\mathcal{Z}(q)$ at ξ is not contained in the tangent space to S at ξ.

PARTIAL RESULT FOR $t = -3$

It is easy to see that the membership $q \in \mathcal{G}_d$ is equivalent to the condition that the strict inequality

$$0 < |(Rq)(\xi)| < |(\partial q)(\xi)|$$

holds for every $\xi \in S \cap \mathcal{Z}(q)$, where $\partial q = (\partial_1 q, \ldots, \partial_d q)$, and $|(\partial q)(\xi)|$ is the Euclidian length of the vector $(\partial q)(\xi)$.

PARTIAL RESULT FOR $t = -3$

Here is what we can prove in the case $t = -3$:

Theorem (F.-Xia preprint)

If $q \in \mathcal{G}_d$, $d \geq 3$, then the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is p-essentially normal for every $p > d$.

In the case $d = 3$, we have $\mathcal{H}^{(-3)} = H^2_3$, the Drury-Arveson case in three variables. Therefore the above implies

Corollary

If $q \in \mathcal{G}_3$, then the submodule $[q]$ of H^2_3 is p-essentially normal for every $p > 3$.

What to do next

Question: Suppose that $d \geq 3$. For arbitrary $q \in [z_1, \ldots, z_d]$ and $-d \leq t \leq -3$, is the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ p-essentially normal for $p > d$?
Actually the case $t = -3$ rests on the following explicit, and seemingly rather tractable, analytical problem:

Question: Suppose that $d \geq 3$ and let $q \in [z_1, \ldots, z_d]$. Let R denote the radial derivative. Dose there exist a constant $0 < C = C(q) < \infty$ such that

$$\int |(Rq)(z)f(z)|^2 dv(z) \leq C\|qf\|_{-2}^2$$

for every $f \in [z_1, \ldots, z_d]$?

von Neumann Inequality revisted

A Hierarchy of von Neumann Inequalities?

One might interpret the von Neumann inequality

$$\|p(A_1, \dots, A_d)\| \le \|p\|_{\mathcal{M}} = \|p(M_{z_1}, \dots, M_{z_d})\|$$

as saying that the tuple $(M_{z_1}, \dots, M_{z_d})$ on the Drury-Arveson space H^2_d "dominates" every commuting row contraction $(A_1, \dots, A_d)$.

Given two row contractions $(A_1, \dots, A_d)$ and $(B_1, \dots, B_d)$, it seems fair to say that $(B_1, \dots, B_d)$ dominates $(A_1, \dots, A_d)$ if

$$\|p(A_1, \dots, A_d)\| \le \|p(B_1, \dots, B_d)\|$$

for every $p \in \mathbf{C}[z_1, \dots, z_d]$.

A Hierarchy of von Neumann Inequalities?

Or, perhaps one can relax this condition slightly:

If there is a constant $0 < C < \infty$ such that

$$\|p(A_1, \ldots, A_d)\| \leq C\|p(B_1, \ldots, B_d)\|$$

for every $p \in \mathbf{C}[z_1, \ldots, z_d]$, one might still say that the tuple $(B_1, \ldots, B_d)$ dominates tuple $(A_1, \ldots, A_d)$.

A Hierarchy of von Neumann Inequalities?

The main point is this: we can ask the rather restricted question whether a given tuple $(B_1, \ldots, B_d)$ dominates (whatever the word means) a particular $(A_1, \ldots, A_d)$, not the question whether it dominates a general class of $(A_1, \ldots, A_d)$'s.

In other words, the tuple $(B_1, \ldots, B_d)$ may not be as dominating as the tuple $(M_{z_1}, \ldots, M_{z_d})$ on H^2_d, but does it dominate $(A_1, \ldots, A_d)$ nonetheless?

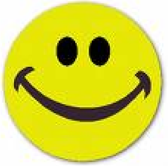
Obviously, we are thinking about some sort of hierarchy, albeit partial, among commuting tuple of operators.

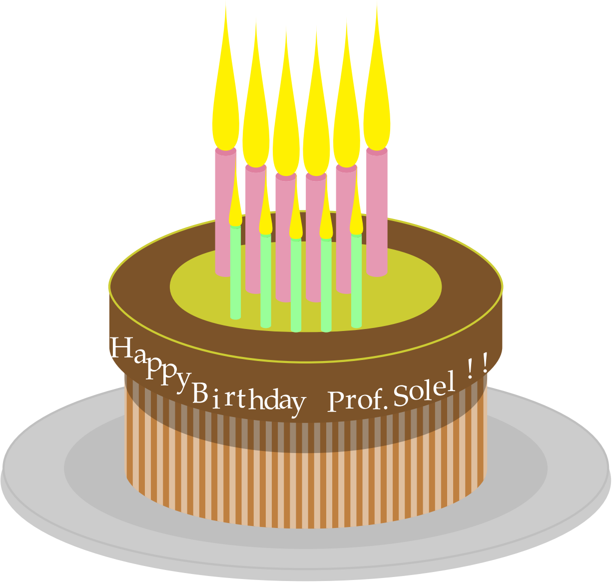
A Hierarchy of von Neumann Inequalities?

F. and Xia. (2014): We give some non-trivial examples of such a hierarchy among commuting tuples of operators.

Question. How about a general theory? Is there a noncommutative version?

Thanks for your attention!




Happy Birthday Prof. Solel !!