

Classification of C^* -envelopes of tensor algebras arising from stochastic matrices

Daniel Markiewicz (Ben-Gurion Univ. of the Negev)

Joint Work with

Adam Dor-On (Univ. of Waterloo)

Multivariable Operator Theory at the Technion
On the occasion of Baruch Solel's 65th Birthday
Technion, June 2017

Dor-On-M.'16 Adam Dor-On and Daniel Markiewicz,
“C*-envelopes of tensor algebras arising from stochastic matrices”, *Integral Equations and Operator Theory* (2017),
doi:10.1007/s00020-017-2382-x (also in the arXiv).

General Problem

What is the C*-envelope of the Tensor Algebra of the subproduct system over \mathbb{N} arising from a stochastic matrix?

There are some surprises when compared to the situation of *product* systems over \mathbb{N} .

Dor-On-M.'16 Adam Dor-On and Daniel Markiewicz,
“C*-envelopes of tensor algebras arising from stochastic matrices”, *Integral Equations and Operator Theory* (2017),
doi:10.1007/s00020-017-2382-x (also in the arXiv).

General Problem

What is the C*-envelope of the Tensor Algebra of the subproduct system over \mathbb{N} arising from a stochastic matrix?

There are some surprises when compared to the situation of *product* systems over \mathbb{N} .

Definition (Shalit-Solel '09, Bhat-Mukherjee '10)

Let M be a vN algebra, let $X = (X_n)_{n \in \mathbb{N}}$ be a family of W^* -correspondences over M , and let $U = (U_{m,n} : X_m \otimes X_n \rightarrow X_{m+n})$ be a family of bounded M -linear maps. We say that X is a **subproduct system over M** if for all $m, n, p \in \mathbb{N}$,

- 1 $X_0 = M$
- 2 $U_{m,n}$ is **co-isometric**
- 3 The family U “behaves like multiplication”: $U_{m,0}$ and $U_{0,n}$ are the right/left multiplications and

$$U_{m+n,p}(U_{m,n} \otimes I_p) = U_{m,n+p}(I_m \otimes U_{n,p})$$

When $U_{m,n}$ is unitary for all m, n we say that X is a **product system**.

Theorem (Muhly-Solel '02, Solel-Shalit '09)

Let M be a vN algebra. Suppose that $\theta : M \rightarrow M$ is a unital normal CP map. Then there exists a canonical subproduct system structure on the family of Arveson-Stinespring correspondences associated to $(\theta^n)_{n \in \mathbb{N}}$.

Definition

Given a countable (possibly infinite) set Ω , a **stochastic matrix over Ω** is a function $P : \Omega \times \Omega \rightarrow \mathbb{R}$ such that $P_{ij} \geq 0$ for all i, j and $\sum_{j \in \Omega} P_{ij} = 1$ for all i .

Subproduct system of a stochastic matrix

There is a **1-1 correspondence between ucp maps of $\ell^\infty(\Omega)$ into itself and stochastic matrices over Ω** given by

$$\theta_P(f)(i) = \sum_{j \in \Omega} P_{ij} f(j)$$

Hence, a stochastic P gives rise to a canonical subproduct system $Arv(P)$.

Theorem (Muhly-Solel '02, Solel-Shalit '09)

Let M be a vN algebra. Suppose that $\theta : M \rightarrow M$ is a unital normal CP map. Then there exists a canonical subproduct system structure on the family of Arveson-Stinespring correspondences associated to $(\theta^n)_{n \in \mathbb{N}}$.

Definition

Given a countable (possibly infinite) set Ω , a **stochastic matrix over Ω** is a function $P : \Omega \times \Omega \rightarrow \mathbb{R}$ such that $P_{ij} \geq 0$ for all i, j and $\sum_{j \in \Omega} P_{ij} = 1$ for all i .

Subproduct system of a stochastic matrix

There is a **1-1 correspondence between ucp maps of $\ell^\infty(\Omega)$ into itself and stochastic matrices over Ω** given by

$$\theta_P(f)(i) = \sum_{j \in \Omega} P_{ij} f(j)$$

Hence, a stochastic P gives rise to a canonical subproduct system $Arv(P)$.

Theorem (Muhly-Solel '02, Solel-Shalit '09)

Let M be a vN algebra. Suppose that $\theta : M \rightarrow M$ is a unital normal CP map. Then there exists a canonical subproduct system structure on the family of Arveson-Stinespring correspondences associated to $(\theta^n)_{n \in \mathbb{N}}$.

Definition

Given a countable (possibly infinite) set Ω , a **stochastic matrix over Ω** is a function $P : \Omega \times \Omega \rightarrow \mathbb{R}$ such that $P_{ij} \geq 0$ for all i, j and $\sum_{j \in \Omega} P_{ij} = 1$ for all i .

Subproduct system of a stochastic matrix

There is a **1-1 correspondence between ucp maps of $\ell^\infty(\Omega)$** into itself and stochastic matrices over Ω given by

$$\theta_P(f)(i) = \sum_{j \in \Omega} P_{ij} f(j)$$

Hence, a stochastic P gives rise to a canonical subproduct system $Arv(P)$.

Given a subproduct system (X, U) , we define the Fock W^* -correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator

$$S_{\xi}^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n$$

- Tensor algebra (not self-adjoint):

$$\mathcal{T}_+(X) = \overline{\text{Alg}}^{\|\cdot\|} M \cup \{S_{\xi}^{(m)} \mid \forall \xi \in X_m, \forall m\}$$

- Toeplitz algebra: $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$
- Cuntz-Pimsner algebra: $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$ for appropriate $\mathcal{J}(X)$

For the case of subproduct systems, Viselter '12 defined the ideal $\mathcal{J}(X)$ as follows: let Q_n denote the orthogonal projection onto the n^{th} summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n \rightarrow \infty} \|TQ_n\| = 0\}.$$

Given a subproduct system (X, U) , we define the Fock W^* -correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator

$$S_{\xi}^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n$$

- Tensor algebra (not self-adjoint):

$$\mathcal{T}_+(X) = \overline{\text{Alg}^{\|\cdot\|} M \cup \{S_{\xi}^{(m)} \mid \forall \xi \in X_m, \forall m\}}$$

- Toeplitz algebra: $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$
- Cuntz-Pimsner algebra: $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$ for appropriate $\mathcal{J}(X)$

For the case of subproduct systems, Viselter '12 defined the ideal $\mathcal{J}(X)$ as follows: let Q_n denote the orthogonal projection onto the n^{th} summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n \rightarrow \infty} \|TQ_n\| = 0\}.$$

Given a subproduct system (X, U) , we define the Fock W^* -correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator

$$S_{\xi}^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n$$

- **Tensor algebra** (not self-adjoint):

$$\mathcal{T}_+(X) = \overline{\text{Alg}^{\|\cdot\|} M \cup \{S_{\xi}^{(m)} \mid \forall \xi \in X_m, \forall m\}}$$

- **Toeplitz algebra**: $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$
- **Cuntz-Pimsner algebra**: $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$ for appropriate $\mathcal{J}(X)$

For the case of subproduct systems, Viselter '12 defined the ideal $\mathcal{J}(X)$ as follows: let Q_n denote the orthogonal projection onto the n^{th} summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n \rightarrow \infty} \|TQ_n\| = 0\}.$$

Given a subproduct system (X, U) , we define the Fock W^* -correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator

$$S_{\xi}^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n$$

- **Tensor algebra** (not self-adjoint):

$$\mathcal{T}_+(X) = \overline{\text{Alg}^{\|\cdot\|} M \cup \{S_{\xi}^{(m)} \mid \forall \xi \in X_m, \forall m\}}$$

- **Toeplitz algebra:** $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$

- **Cuntz-Pimsner algebra:** $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$ for appropriate $\mathcal{J}(X)$

For the case of subproduct systems, Viselter '12 defined the ideal $\mathcal{J}(X)$ as follows: let Q_n denote the orthogonal projection onto the n^{th} summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n \rightarrow \infty} \|TQ_n\| = 0\}.$$

Given a subproduct system (X, U) , we define the Fock W^* -correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator

$$S_{\xi}^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n$$

- **Tensor algebra** (not self-adjoint):

$$\mathcal{T}_+(X) = \overline{\text{Alg}^{\|\cdot\|} M \cup \{S_{\xi}^{(m)} \mid \forall \xi \in X_m, \forall m\}}$$

- **Toeplitz algebra**: $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$
- **Cuntz-Pimsner algebra**: $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$ for appropriate $\mathcal{J}(X)$

For the case of subproduct systems, Viselter '12 defined the ideal $\mathcal{J}(X)$ as follows: let Q_n denote the orthogonal projection onto the n^{th} summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n \rightarrow \infty} \|TQ_n\| = 0\}.$$

Given a subproduct system (X, U) , we define the Fock W^* -correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator

$$S_{\xi}^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n$$

- **Tensor algebra** (not self-adjoint):

$$\mathcal{T}_+(X) = \overline{\text{Alg}^{\|\cdot\|} M \cup \{S_{\xi}^{(m)} \mid \forall \xi \in X_m, \forall m\}}$$

- **Toeplitz algebra**: $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$
- **Cuntz-Pimsner algebra**: $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$ for appropriate $\mathcal{J}(X)$

For the case of subproduct systems, Viselter '12 defined the ideal $\mathcal{J}(X)$ as follows: let Q_n denote the orthogonal projection onto the n^{th} summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n \rightarrow \infty} \|TQ_n\| = 0\}.$$

Example (Product system $\mathcal{P}^{\mathbb{C}}$)

Let $X = \mathcal{P}^{\mathbb{C}} = \bigcup_{n \in \mathbb{N}} \mathbb{C}$ be the “line bundle” product system.

- We have $\mathcal{F}_X = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathcal{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathcal{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system \mathcal{P}_E is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

Example (Product system $\mathcal{P}^{\mathbb{C}}$)

Let $X = \mathcal{P}^{\mathbb{C}} = \bigcup_{n \in \mathbb{N}} \mathbb{C}$ be the “line bundle” product system.

- We have $\mathcal{F}_X = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathcal{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathcal{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system \mathcal{P}_E is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

Example (Product system $\mathcal{P}^{\mathbb{C}}$)

Let $X = \mathcal{P}^{\mathbb{C}} = \bigcup_{n \in \mathbb{N}} \mathbb{C}$ be the “line bundle” product system.

- We have $\mathcal{F}_X = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathcal{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathcal{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system \mathcal{P}_E is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

Example (Product system $\mathcal{P}^{\mathbb{C}}$)

Let $X = \mathcal{P}^{\mathbb{C}} = \cup_{n \in \mathbb{N}} \mathbb{C}$ be the “line bundle” product system.

- We have $\mathcal{F}_X = \oplus_{n \in \mathbb{N}} \mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathcal{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathcal{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system \mathcal{P}_E is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

Example (Product system $\mathcal{P}^{\mathbb{C}}$)

Let $X = \mathcal{P}^{\mathbb{C}} = \cup_{n \in \mathbb{N}} \mathbb{C}$ be the “line bundle” product system.

- We have $\mathcal{F}_X = \oplus_{n \in \mathbb{N}} \mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathcal{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathcal{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system \mathcal{P}_E is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

Example (Product system $\mathcal{P}^{\mathbb{C}}$)

Let $X = \mathcal{P}^{\mathbb{C}} = \bigcup_{n \in \mathbb{N}} \mathbb{C}$ be the “line bundle” product system.

- We have $\mathcal{F}_X = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathcal{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathcal{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system \mathcal{P}_E is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

Example (Product system $\mathcal{P}^{\mathbb{C}}$)

Let $X = \mathcal{P}^{\mathbb{C}} = \bigcup_{n \in \mathbb{N}} \mathbb{C}$ be the “line bundle” product system.

- We have $\mathcal{F}_X = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift.
- $\mathcal{T}_+(\mathcal{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$ the disk algebra
- $\mathcal{T}(\mathcal{P}^{\mathbb{C}})$ is the original Toeplitz algebra
- $\mathcal{O}(\mathcal{P}^{\mathbb{C}}) = C(\mathbb{T})$

Theorem (Viselter '12)

If E is a correspondence and its associated product system \mathcal{P}_E is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences (via the associated product system).

In a previous paper with A. Dor-On, we studied the tensor algebras in their own right. Let's do a quick review.

- Recall that a stochastic matrix P is **essential** if for every i , $P_{ij}^n > 0$ for some n implies that $\exists m$ such that $P_{ji}^m > 0$.
- The **support** of P is the matrix $\text{supp}(P)$ given by

$$\text{supp}(P)_{ij} = \begin{cases} 1, & P_{ij} \neq 0 \\ 0, & P_{ij} = 0 \end{cases}$$

Theorem (Dor-On-M.'14)

Let P and Q be **finite** stochastic matrices over Ω . TFAE:

- There is an **algebraic** isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- there is a graded **comp. bounded** isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- $\text{Arv}(P)$ and $\text{Arv}(Q)$ are **similar** up to change of base

Furthermore, if P and Q are **essential**, those conditions hold if and only if P and Q have the same **supports** up to permutation of Ω .

In a previous paper with A. Dor-On, we studied the tensor algebras in their own right. Let's do a quick review.

- Recall that a stochastic matrix P is **essential** if for every i , $P_{ij}^n > 0$ for some n implies that $\exists m$ such that $P_{ji}^m > 0$.
- The **support** of P is the matrix $\text{supp}(P)$ given by

$$\text{supp}(P)_{ij} = \begin{cases} 1, & P_{ij} \neq 0 \\ 0, & P_{ij} = 0 \end{cases}$$

Theorem (Dor-On-M.'14)

Let P and Q be *finite* stochastic matrices over Ω . TFAE:

- 1 There is an **algebraic** isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 2 there is a graded **comp. bounded** isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 3 $\text{Arv}(P)$ and $\text{Arv}(Q)$ are **similar** up to change of base

Furthermore, if P and Q are **essential**, those conditions hold if and only if P and Q have the same **supports** up to permutation of Ω .

In a previous paper with A. Dor-On, we studied the tensor algebras in their own right. Let's do a quick review.

- Recall that a stochastic matrix P is **essential** if for every i , $P_{ij}^n > 0$ for some n implies that $\exists m$ such that $P_{ji}^m > 0$.
- The **support of P** is the matrix $\text{supp}(P)$ given by

$$\text{supp}(P)_{ij} = \begin{cases} 1, & P_{ij} \neq 0 \\ 0, & P_{ij} = 0 \end{cases}$$

Theorem (Dor-On-M.'14)

Let P and Q be *finite* stochastic matrices over Ω . TFAE:

- 1 There is an *algebraic* isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 2 there is a graded *comp. bounded* isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 3 $\text{Arv}(P)$ and $\text{Arv}(Q)$ are *similar* up to change of base

Furthermore, if P and Q are *essential*, those conditions hold if and only if P and Q have the same *supports* up to permutation of Ω .

In a previous paper with A. Dor-On, we studied the tensor algebras in their own right. Let's do a quick review.

- Recall that a stochastic matrix P is **essential** if for every i , $P_{ij}^n > 0$ for some n implies that $\exists m$ such that $P_{ji}^m > 0$.
- The **support of P** is the matrix $\text{supp}(P)$ given by

$$\text{supp}(P)_{ij} = \begin{cases} 1, & P_{ij} \neq 0 \\ 0, & P_{ij} = 0 \end{cases}$$

Theorem (Dor-On-M.'14)

Let P and Q be **finite** stochastic matrices over Ω . TFAE:

- 1 There is an **algebraic** isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 2 there is a graded **comp. bounded** isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 3 $\text{Arv}(P)$ and $\text{Arv}(Q)$ are **similar** up to change of base

Furthermore, if P and Q are **essential**, those conditions hold if and only if P and Q have the same **supports** up to permutation of Ω .

- A stochastic matrix P is **recurrent** if $\sum_n (P^n)_{ii} = \infty$ for all i .

Theorem (Dor-On-M.'14)

Let P and Q be stochastic matrices over Ω . TFAE:

- 1 There is an **isometric** isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 2 there is a **graded comp. isometric** isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.
- 3 $\text{Arv}(P)$ and $\text{Arv}(Q)$ are **unitarily isomorphic** up to change of base.

Furthermore, if P and Q are **recurrent**, those conditions hold if and only if P and Q are the same up to **permutation** of Ω .

We also computed the Cuntz-Pimsner algebra in the sense of Viselter.

Theorem (Dor-On-M.'14)

If P is irreducible $d \times d$ stochastic, then $\mathcal{O}(P) \simeq C(\mathbb{T}) \otimes M_d(\mathbb{C})$.

We thank Dilian Yang for pointing out a gap, fixed in Dor-On-M.'16.

We will turn the uncomplicated nature of $\mathcal{O}(P)$ to our advantage to study the C^* -envelope of $\mathcal{T}_+(P)$.

We also computed the Cuntz-Pimsner algebra in the sense of Viselter.

Theorem (Dor-On-M.'14)

If P is irreducible $d \times d$ stochastic, then $\mathcal{O}(P) \simeq C(\mathbb{T}) \otimes M_d(\mathbb{C})$.

We thank Dilian Yang for pointing out a gap, fixed in Dor-On-M.'16.

We will turn the uncomplicated nature of $\mathcal{O}(P)$ to our advantage to study the C^* -envelope of $\mathcal{T}_+(P)$.

We also computed the Cuntz-Pimsner algebra in the sense of Viselter.

Theorem (Dor-On-M.'14)

If P is irreducible $d \times d$ stochastic, then $\mathcal{O}(P) \simeq C(\mathbb{T}) \otimes M_d(\mathbb{C})$.

We thank Dilian Yang for pointing out a gap, fixed in Dor-On-M.'16.

We will turn the uncomplicated nature of $\mathcal{O}(P)$ to our advantage to study the C^* -envelope of $\mathcal{T}_+(P)$.

Definition (C*-envelope - existence proved by Hamana '79)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra. The **C*-envelope** of \mathcal{A} consists of a C*-algebra $C_{\text{env}}^*(\mathcal{A})$ and a comp. isometric embedding $\iota : \mathcal{A} \rightarrow C_{\text{env}}^*(\mathcal{A})$ with the following universal property: if $j : \mathcal{A} \rightarrow B$ is a comp. isometric embedding and $B = C^*(j(\mathcal{A}))$, then there is a *-homomorphism $\phi : B \rightarrow C_{\text{env}}^*(\mathcal{A})$ such that $\phi(j(a)) = \iota(a)$ for all $a \in \mathcal{A}$.

Definition (Arveson '69)

Let \mathcal{S} be an operator system. We say that a UCP map $\phi : \mathcal{S} \rightarrow B(H)$ has the **unique extension property (UEP)** if it has a unique cp extension $\tilde{\phi} : C^*(\mathcal{S}) \rightarrow B(H)$ which is a *-rep. If $\tilde{\phi}$ is irreducible, then ϕ is called a **boundary representation** of \mathcal{S} .

Theorem (Arveson '08 for \mathcal{A} separable, Davidson-Kennedy '13)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra and let $S = \mathcal{A} + \mathcal{A}^*$. Let π be the direct sum of all boundary representations of \mathcal{A} . Then the C*-envelope of \mathcal{A} is given by the pair $\pi \upharpoonright_{\mathcal{A}}$ and $C^*(\pi(S))$.

Definition (C*-envelope - existence proved by Hamana '79)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra. The **C*-envelope** of \mathcal{A} consists of a C*-algebra $C_{\text{env}}^*(\mathcal{A})$ and a comp. isometric embedding $\iota : \mathcal{A} \rightarrow C_{\text{env}}^*(\mathcal{A})$ with the following universal property: if $j : \mathcal{A} \rightarrow B$ is a comp. isometric embedding and $B = C^*(j(\mathcal{A}))$, then there is a *-homomorphism $\phi : B \rightarrow C_{\text{env}}^*(\mathcal{A})$ such that $\phi(j(a)) = \iota(a)$ for all $a \in \mathcal{A}$.

Definition (Arveson '69)

Let \mathcal{S} be an operator system. We say that a UCP map $\phi : \mathcal{S} \rightarrow B(H)$ has the **unique extension property (UEP)** if it has a unique cp extension $\tilde{\phi} : C^*(\mathcal{S}) \rightarrow B(H)$ which is a *-rep. If $\tilde{\phi}$ is irreducible, then ϕ is called a **boundary representation** of \mathcal{S} .

Theorem (Arveson '08 for \mathcal{A} separable, Davidson-Kennedy '13)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra and let $S = \mathcal{A} + \mathcal{A}^$. Let π be the direct sum of all boundary representations of \mathcal{A} . Then the C*-envelope of \mathcal{A} is given by the pair $\pi \upharpoonright_{\mathcal{A}}$ and $C^*(\pi(S))$.*

Definition (C*-envelope - existence proved by Hamana '79)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra. The **C*-envelope** of \mathcal{A} consists of a C*-algebra $C_{\text{env}}^*(\mathcal{A})$ and a comp. isometric embedding $\iota : \mathcal{A} \rightarrow C_{\text{env}}^*(\mathcal{A})$ with the following universal property: if $j : \mathcal{A} \rightarrow B$ is a comp. isometric embedding and $B = C^*(j(\mathcal{A}))$, then there is a *-homomorphism $\phi : B \rightarrow C_{\text{env}}^*(\mathcal{A})$ such that $\phi(j(a)) = \iota(a)$ for all $a \in \mathcal{A}$.

Definition (Arveson '69)

Let \mathcal{S} be an operator system. We say that a UCP map $\phi : \mathcal{S} \rightarrow B(H)$ has the **unique extension property (UEP)** if it has a unique cp extension $\tilde{\phi} : C^*(\mathcal{S}) \rightarrow B(H)$ which is a *-rep. If $\tilde{\phi}$ is irreducible, then ϕ is called a **boundary representation** of \mathcal{S} .

Theorem (Arveson '08 for \mathcal{A} separable, Davidson-Kennedy '13)

Let $\mathcal{A} \subseteq B(H)$ be a unital closed subalgebra and let $S = \mathcal{A} + \mathcal{A}^$. Let π be the direct sum of all boundary representations of \mathcal{A} . Then the C*-envelope of \mathcal{A} is given by the pair $\pi \upharpoonright_{\mathcal{A}}$ and $C^*(\pi(S))$.*

Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C*-correspondence, then $C_{\text{env}}^*(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit '11)

If X is a *commutative* subproduct system of fin. dim. Hilbert space fibers, then $C_{\text{env}}^*(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit '15)

If X is a subproduct system of fin. dim. Hilbert space fibers *arising from a subshift of finite type*, then $C_{\text{env}}^*(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

- So far, this seemed to suggest a dichotomy.
- In all these examples, however, X was either product system or was composed of Hilbert spaces.
- First candidate outside that context: **stochastic matrices**.

Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C*-correspondence, then $C_{\text{env}}^*(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit '11)

If X is a *commutative* subproduct system of fin. dim. Hilbert space fibers, then $C_{\text{env}}^*(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit '15)

If X is a subproduct system of fin. dim. Hilbert space fibers *arising from a subshift of finite type*, then $C_{\text{env}}^*(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

- So far, this seemed to suggest a dichotomy.
- In all these examples, however, X was either product system or was composed of Hilbert spaces.
- First candidate outside that context: *stochastic matrices*.

Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C*-correspondence, then $C_{\text{env}}^*(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit '11)

If X is a **commutative** subproduct system of fin. dim. Hilbert space fibers, then $C_{\text{env}}^*(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit '15)

If X is a subproduct system of fin. dim. Hilbert space fibers **arising from a subshift of finite type**, then $C_{\text{env}}^*(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

- So far, this seemed to suggest a dichotomy.
- In all these examples, however, X was either product system or was composed of Hilbert spaces.
- First candidate outside that context: **stochastic matrices**.

Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C*-correspondence, then $C_{\text{env}}^*(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit '11)

If X is a *commutative* subproduct system of fin. dim. Hilbert space fibers, then $C_{\text{env}}^*(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit '15)

If X is a subproduct system of fin. dim. Hilbert space fibers *arising from a subshift of finite type*, then $C_{\text{env}}^*(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

- So far, this seemed to suggest a dichotomy.
- In all these examples, however, X was either product system or was composed of Hilbert spaces.
- First candidate outside that context: *stochastic matrices*.

Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C*-correspondence, then $C_{\text{env}}^*(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit '11)

If X is a *commutative* subproduct system of fin. dim. Hilbert space fibers, then $C_{\text{env}}^*(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit '15)

If X is a subproduct system of fin. dim. Hilbert space fibers *arising from a subshift of finite type*, then $C_{\text{env}}^*(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

- So far, this seemed to suggest a dichotomy.
- In all these examples, however, X was either product system or was composed of Hilbert spaces.
- First candidate outside that context: *stochastic matrices*.

Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C*-correspondence, then $C_{\text{env}}^*(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit '11)

If X is a *commutative* subproduct system of fin. dim. Hilbert space fibers, then $C_{\text{env}}^*(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit '15)

If X is a subproduct system of fin. dim. Hilbert space fibers *arising from a subshift of finite type*, then $C_{\text{env}}^*(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

- So far, this seemed to suggest a dichotomy.
- In all these examples, however, X was either product system or was composed of Hilbert spaces.
- First candidate outside that context: *stochastic matrices*.

Q: What is the C*-envelope of a tensor algebra?

Theorem (From Muhly-Solel '98 (...) to Katsoulis and Kribs '06)

If E is a C*-correspondence, then $C_{\text{env}}^*(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit '11)

If X is a *commutative* subproduct system of fin. dim. Hilbert space fibers, then $C_{\text{env}}^*(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit '15)

If X is a subproduct system of fin. dim. Hilbert space fibers *arising from a subshift of finite type*, then $C_{\text{env}}^*(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

- So far, this seemed to suggest a dichotomy.
- In all these examples, however, X was either product system or was composed of Hilbert spaces.
- First candidate outside that context: [stochastic matrices](#).

- Recall if P is irreducible finite stochastic, $\mathcal{O}(P) \simeq C(\mathbb{T}) \otimes M_d(\mathbb{C})$.
- Let $H = \mathcal{F}_{\text{Arv}(P)} \otimes \ell^2(\Omega)$. We have a canonical representation $\pi : \mathcal{T}(P) \rightarrow B(H)$ which breaks up into d subrepresentations π_k on the “column-like” spaces $H_k = \mathcal{F}_{\text{Arv}(P)} \otimes \mathbb{C}e_k$.

Theorem (Dor-On-M.'16)

If P is irreducible $d \times d$ stochastic, then $\mathcal{J}(\mathcal{T}(P)) \simeq \bigoplus_{j=1}^d \mathbb{K}(H_j)$.
Therefore we have an exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^d \mathbb{K}(H_j) \longrightarrow \mathcal{T}(P) \longrightarrow C(\mathbb{T}) \otimes M_d(\mathbb{C}) \longrightarrow 0$$

Moreover, all irreducible representations of $\mathcal{T}(P)$ are unitarily equivalent to appropriate π_k or arise from the point evaluations on \mathbb{T} .

- Recall if P is irreducible finite stochastic, $\mathcal{O}(P) \simeq C(\mathbb{T}) \otimes M_d(\mathbb{C})$.
- Let $H = \mathcal{F}_{Arv(P)} \otimes \ell^2(\Omega)$. We have a canonical representation $\pi : \mathcal{T}(P) \rightarrow B(H)$ which breaks up into d subrepresentations π_k on the “column-like” spaces $H_k = \mathcal{F}_{Arv(P)} \otimes \mathbb{C}e_k$.

Theorem (Dor-On-M.'16)

If P is irreducible $d \times d$ stochastic, then $\mathcal{J}(\mathcal{T}(P)) \simeq \bigoplus_{j=1}^d \mathbb{K}(H_j)$.
Therefore we have an exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^d \mathbb{K}(H_j) \longrightarrow \mathcal{T}(P) \longrightarrow C(\mathbb{T}) \otimes M_d(\mathbb{C}) \longrightarrow 0$$

Moreover, all irreducible representations of $\mathcal{T}(P)$ are unitarily equivalent to appropriate π_k or arise from the point evaluations on \mathbb{T} .

Theorem (Dor-On-M.'16)

Suppose that P is an irreducible matrix of size d . The point evaluations of $C(\mathbb{T}) \otimes M_d(\mathbb{C})$ lift to boundary representations of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$. Therefore have an exact sequence

$$0 \longrightarrow \bigoplus_{j \in \Omega_b^P} \mathbb{K}(H_j) \longrightarrow C_{\text{env}}^*(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_d \longrightarrow 0$$

where Ω_b^P is the set of states k for which π_k is boundary.

Definition

Let P be an irreducible r -periodic stochastic matrix of size d . A state $k \in \Omega$ is called **exclusive** if whenever for $i \in \Omega$ and $n \in \mathbb{N}$ we have $P_{ik}^{(n)} > 0$, then $P_{ik}^{(n)} = 1$.

We say that P has the **multiple-arrival property** if whenever $k, s \in \Omega$ are distinct non-exclusive states such that whenever k leads to s in n steps, then there exists $k' \neq k \in \Omega$ such that k' leads to s in n steps.

Example

If P is r -periodic, then by permuting states it has the cyclic block decomposition

$$\begin{bmatrix} 0 & P_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{r-2} \\ P_{r-1} & \cdots & 0 & 0 \end{bmatrix}, \quad \text{example: } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

If such a matrix has **full-support**, which is to say no zeros in the blocks P_j , then it has multiple-arrival.

Definition

Let P be an irreducible r -periodic stochastic matrix of size d . A state $k \in \Omega$ is called **exclusive** if whenever for $i \in \Omega$ and $n \in \mathbb{N}$ we have $P_{ik}^{(n)} > 0$, then $P_{ik}^{(n)} = 1$.

We say that P has the **multiple-arrival property** if whenever $k, s \in \Omega$ are distinct non-exclusive states such that whenever k leads to s in n steps, then there exists $k' \neq k \in \Omega$ such that k' leads to s in n steps.

Example

If P is r -periodic, then by permuting states it has the cyclic block decomposition

$$\begin{bmatrix} 0 & P_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{r-2} \\ P_{r-1} & \cdots & 0 & 0 \end{bmatrix}, \quad \text{example: } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

If such a matrix has **full-support**, which is to say no zeros in the blocks P_j , then it has multiple-arrival.

Theorem (Dor-On-M.'16)

Let P be an irreducible finite stochastic matrix. If $k \in \Omega$ is exclusive, then π_k is not a boundary rep.

Theorem (Dor-On-M.'16)

Suppose that P is a finite irreducible matrix with multiple-arrival. Then π_k is a boundary representation if and only if k is non-exclusive. Therefore, the C*-envelope of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$ corresponds to the quotient by the ideal

$$\bigcap_{k \text{ non-exclusive}} \{ T \in \mathcal{J}(P) \mid \pi_k(T) = 0 \} \simeq_{\pi} \bigoplus_{j \text{ exclusive}} \mathbb{K}(H_j)$$

Thus we have an exact sequence

$$0 \longrightarrow \bigoplus_{j \text{ non-exclusive}} \mathbb{K}(H_j) \longrightarrow C_{\text{env}}^*(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_d \longrightarrow 0$$

Theorem (Dor-On-M.'16)

Let P be an irreducible finite stochastic matrix. If $k \in \Omega$ is exclusive, then π_k is not a boundary rep.

Theorem (Dor-On-M.'16)

Suppose that P is a finite irreducible matrix with multiple-arrival. Then π_k is a boundary representation if and only if k is non-exclusive. Therefore, the C*-envelope of $\mathcal{T}_+(P)$ inside $\mathcal{T}(P)$ corresponds to the quotient by the ideal

$$\bigcap_{k \text{ non-exclusive}} \{ T \in \mathcal{J}(P) \mid \pi_k(T) = 0 \} \simeq_{\pi} \bigoplus_{j \text{ exclusive}} \mathbb{K}(H_j)$$

Thus we have an exact sequence

$$0 \longrightarrow \bigoplus_{j \text{ non-exclusive}} \mathbb{K}(H_j) \longrightarrow C_{\text{env}}^*(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_d \longrightarrow 0$$

Theorem (Dor-On-M.'16)

Let P be an irreducible stochastic *finite* matrix with multiple-arrival.

- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{T}(P)$ iff all states non-exclusive.
- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{O}(P)$ iff all states exclusive.

Example (Dor-On-M.'16: Dichotomy fails)

$C_{\text{env}}^*(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ and $\mathcal{O}(P)$ are all different for $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$.

Since P is 2-periodic, we see from its cyclic decomposition it has full-support. Therefore it has the multiple-arrival property. The only exclusive column is $k = 3$. Therefore we have an exact sequence

$$0 \longrightarrow \mathbb{K}(H_1) \oplus \mathbb{K}(H_2) \longrightarrow C_{\text{env}}^*(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_3 \longrightarrow 0$$

Theorem (Dor-On-M.'16)

Let P be an irreducible stochastic *finite* matrix with multiple-arrival.

- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{T}(P)$ iff all states non-exclusive.
- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{O}(P)$ iff all states exclusive.

Example (Dor-On-M.'16: Dichotomy fails)

$C_{\text{env}}^*(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ and $\mathcal{O}(P)$ are all different for $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$.

Since P is 2-periodic, we see from its cyclic decomposition it has full-support. Therefore it has the multiple-arrival property. The only exclusive column is $k = 3$. Therefore we have an exact sequence

$$0 \longrightarrow \mathbb{K}(H_1) \oplus \mathbb{K}(H_2) \longrightarrow C_{\text{env}}^*(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_3 \longrightarrow 0$$

Theorem (Dor-On-M.'16)

Let P be an irreducible stochastic *finite* matrix with multiple-arrival.

- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{T}(P)$ iff all states non-exclusive.
- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{O}(P)$ iff all states exclusive.

Example (Dor-On-M.'16: Dichotomy fails)

$C_{\text{env}}^*(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ and $\mathcal{O}(P)$ are all different for $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$.

Since P is 2-periodic, we see from its cyclic decomposition it has full-support. Therefore it has the multiple-arrival property. The only exclusive column is $k = 3$. Therefore we have an exact sequence

$$0 \longrightarrow \mathbb{K}(H_1) \oplus \mathbb{K}(H_2) \longrightarrow C_{\text{env}}^*(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_3 \longrightarrow 0$$

Theorem (Dor-On-M.'16)

Let P be an irreducible stochastic *finite* matrix with multiple-arrival.

- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{T}(P)$ iff all states non-exclusive.
- $C_{\text{env}}^*(\mathcal{T}_+(P)) \cong \mathcal{O}(P)$ iff all states exclusive.

Example (Dor-On-M.'16: Dichotomy fails)

$C_{\text{env}}^*(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ and $\mathcal{O}(P)$ are all different for $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$.

Since P is 2-periodic, we see from its cyclic decomposition it has full-support. Therefore it has the multiple-arrival property. The only exclusive column is $k = 3$. Therefore we have an exact sequence

$$0 \longrightarrow \mathbb{K}(H_1) \oplus \mathbb{K}(H_2) \longrightarrow C_{\text{env}}^*(\mathcal{T}_+(P)) \longrightarrow C(\mathbb{T}) \otimes M_3 \longrightarrow 0$$

Q: If dichotomy fails, what are the possibilities for $C_{\text{env}}^*(\mathcal{T}_+(P))$?

Recall $\Omega_b^P = \{k \in \Omega : \pi_k \text{ is boundary for } P\}$

Theorem (Dor-On-M.'16)

Let P be a finite irreducible stochastic.

① If P has a non-exclusive state then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}^{|\Omega_b|} \quad \text{and} \quad K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \{0\}$$

② If all states are exclusive then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}$$

Theorem (Dor-On-M.'16)

Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $|\Omega_b^P| = |\Omega_b^Q|$ if and only if $C_{\text{env}}^*(\mathcal{T}_+(P))$ and $C_{\text{env}}^*(\mathcal{T}_+(Q))$ are stably isomorphic.

Q: If dichotomy fails, what are the possibilities for $C_{\text{env}}^*(\mathcal{T}_+(P))$?
 Recall $\Omega_b^P = \{k \in \Omega : \pi_k \text{ is boundary for } P\}$

Theorem (Dor-On-M.'16)

Let P be a finite irreducible stochastic.

① If P has a non-exclusive state then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}^{|\Omega_b|} \quad \text{and} \quad K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \{0\}$$

② If all states are exclusive then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}$$

Theorem (Dor-On-M.'16)

Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $|\Omega_b^P| = |\Omega_b^Q|$ if and only if $C_{\text{env}}^*(\mathcal{T}_+(P))$ and $C_{\text{env}}^*(\mathcal{T}_+(Q))$ are stably isomorphic.

Q: If dichotomy fails, what are the possibilities for $C_{\text{env}}^*(\mathcal{T}_+(P))$?
 Recall $\Omega_b^P = \{k \in \Omega : \pi_k \text{ is boundary for } P\}$

Theorem (Dor-On-M.'16)

Let P be a finite irreducible stochastic.

- ① If P has a non-exclusive state then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}^{|\Omega_b|} \quad \text{and} \quad K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \{0\}$$

- ② If all states are exclusive then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}$$

Theorem (Dor-On-M.'16)

Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $|\Omega_b^P| = |\Omega_b^Q|$ if and only if $C_{\text{env}}^*(\mathcal{T}_+(P))$ and $C_{\text{env}}^*(\mathcal{T}_+(Q))$ are stably isomorphic.

Q: If dichotomy fails, what are the possibilities for $C_{\text{env}}^*(\mathcal{T}_+(P))$?
 Recall $\Omega_b^P = \{k \in \Omega : \pi_k \text{ is boundary for } P\}$

Theorem (Dor-On-M.'16)

Let P be a finite irreducible stochastic.

- ① If P has a non-exclusive state then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}^{|\Omega_b|} \quad \text{and} \quad K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \{0\}$$

- ② If all states are exclusive then

$$K_0(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong K_1(C_{\text{env}}^*(\mathcal{T}_+(P))) \cong \mathbb{Z}$$

Theorem (Dor-On-M.'16)

Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $|\Omega_b^P| = |\Omega_b^Q|$ if and only if $C_{\text{env}}^*(\mathcal{T}_+(P))$ and $C_{\text{env}}^*(\mathcal{T}_+(Q))$ are stably isomorphic.

Definition

Let P be an r -periodic irreducible stochastic matrix over Ω of size d , and $k \in \Omega$. Let $\Omega_0, \dots, \Omega_{r-1}$ be a cyclic decomposition for P , so that $\sigma(k)$ is the unique index such that $k \in \Omega_{\sigma(k)}$. The k -th column nullity of P is

$$\mathcal{N}_P(k) = \sum_{m=1}^{\infty} |\{ i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} = 0 \}|$$

Intuition: It counts the number of zeros in the k^{th} column of the powers of P , relative to the cyclic decomposition support.

$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \rightarrow \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \rightarrow \dots$$

Note the series is actually a sum, because the matrix powers fill-out eventually.

Definition

Let P be an r -periodic irreducible stochastic matrix over Ω of size d , and $k \in \Omega$. Let $\Omega_0, \dots, \Omega_{r-1}$ be a cyclic decomposition for P , so that $\sigma(k)$ is the unique index such that $k \in \Omega_{\sigma(k)}$. The k -th column nullity of P is

$$\mathcal{N}_P(k) = \sum_{m=1}^{\infty} |\{ i \in \Omega_{\sigma(k)-m} \mid P_{ik}^{(m)} = 0 \}|$$

Intuition: It counts the number of zeros in the k^{th} column of the powers of P , relative to the cyclic decomposition support.

$$\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \rightarrow \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \rightarrow \dots$$

Note the series is actually a sum, because the matrix powers fill-out eventually.

Theorem (Dor-On-M.'16)

Let P and Q be finite irreducible stochastic matrices over Ω^P and Ω^Q respectively. Then $C_{\text{env}}^*(\mathcal{T}_+(P))$ and $C_{\text{env}}^*(\mathcal{T}_+(Q))$ are $*$ -isomorphic if and only if

- 1 $|\Omega^P| = |\Omega^Q|$ (let d be this number)
- 2 there is a bijection $\tau : \Omega_b^P \rightarrow \Omega_b^Q$ such that

$$\forall k \in \Omega_b^P, \quad \mathcal{N}_P(k) \equiv \mathcal{N}_Q(\tau(k)) \pmod{d}.$$

Example

Suppose matrices for P, Q, R are stochastic with matrices supported on graphs (so multiple-arrival)

$$Gr(P) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad Gr(Q) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad Gr(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Omega_b^P = \{1, 2\}, \quad \mathcal{N}_P(j) = 0, \quad j = 1, 2, 3$$

$$\Omega_b^Q = \{1, 2, 3\}, \quad \mathcal{N}_Q(j) = 0, \quad j = 1, 2, 3$$

$$\Omega_b^R = \{1, 2, 3\}, \quad \mathcal{N}_R(1) = \mathcal{N}_R(2) = 0, \quad \mathcal{N}_R(3) = 1,$$

Let \cong denote $*$ -isomorphism. Then:

$$C_{\text{env}}^*(\mathcal{T}_+(P)) \otimes \mathbb{K} \not\cong C_{\text{env}}^*(\mathcal{T}_+(Q)) \otimes \mathbb{K} \cong C_{\text{env}}^*(\mathcal{T}_+(R)) \otimes \mathbb{K}$$

$$C_{\text{env}}^*(\mathcal{T}_+(Q)) \not\cong C_{\text{env}}^*(\mathcal{T}_+(R))$$

Example

Suppose matrices for P, Q, R are stochastic with matrices supported on graphs (so multiple-arrival)

$$\text{Gr}(P) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{Gr}(Q) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{Gr}(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Omega_b^P = \{1, 2\}, \quad \mathcal{N}_P(j) = 0, \quad j = 1, 2, 3$$

$$\Omega_b^Q = \{1, 2, 3\}, \quad \mathcal{N}_Q(j) = 0, \quad j = 1, 2, 3$$

$$\Omega_b^R = \{1, 2, 3\}, \quad \mathcal{N}_R(1) = \mathcal{N}_R(2) = 0, \quad \mathcal{N}_R(3) = 1,$$

Let \cong denote *-isomorphism. Then:

$$C_{\text{env}}^*(\mathcal{T}_+(P)) \otimes \mathbb{K} \not\cong C_{\text{env}}^*(\mathcal{T}_+(Q)) \otimes \mathbb{K} \cong C_{\text{env}}^*(\mathcal{T}_+(R)) \otimes \mathbb{K}$$

$$C_{\text{env}}^*(\mathcal{T}_+(Q)) \not\cong C_{\text{env}}^*(\mathcal{T}_+(R))$$

Example

Suppose matrices for P, Q, R are stochastic with matrices supported on graphs (so multiple-arrival)

$$Gr(P) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad Gr(Q) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad Gr(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Omega_b^P = \{1, 2\}, \quad \mathcal{N}_P(j) = 0, \quad j = 1, 2, 3$$

$$\Omega_b^Q = \{1, 2, 3\}, \quad \mathcal{N}_Q(j) = 0, \quad j = 1, 2, 3$$

$$\Omega_b^R = \{1, 2, 3\}, \quad \mathcal{N}_R(1) = \mathcal{N}_R(2) = 0, \quad \mathcal{N}_R(3) = 1,$$

Let \cong denote $*$ -isomorphism. Then:

$$C_{\text{env}}^*(\mathcal{T}_+(P)) \otimes \mathbb{K} \not\cong C_{\text{env}}^*(\mathcal{T}_+(Q)) \otimes \mathbb{K} \cong C_{\text{env}}^*(\mathcal{T}_+(R)) \otimes \mathbb{K}$$

$$C_{\text{env}}^*(\mathcal{T}_+(Q)) \not\cong C_{\text{env}}^*(\mathcal{T}_+(R))$$

Thank you!

Extension theory:

$$0 \rightarrow K \xrightarrow{\iota} A \xrightarrow{\pi} B \rightarrow 0$$

can be studied through Busby invariant $\eta : B \rightarrow Q(K) \cong M(K)/K$, since have $\theta : A \rightarrow M(K)$ by $\theta(a)c = \iota^{-1}(a\iota(c))$

Equivalence of exact sequences gives relation for Busby inv.:

$$\exists \kappa : K_1 \rightarrow K_2 \text{ and } \beta : B_1 \rightarrow B_2 \text{ s.t. } \tilde{\kappa}\eta_1 = \eta_2\beta.$$

In our case closely connected to $K = \mathbb{K}$ for which a lot is known. There is a group structure on the set of equivalence classes of extensions (both weak and strong) since B is nuclear separable (Choi-Effros).

$$\text{Ext}_s(B) \rightarrow \text{Ext}_w(B) \rightarrow \text{Hom}(K_1(B), \mathbb{Z})$$

We use work of Paschke and Salinas, which characterizes this objects when K is a sum of compacts, and sweat to identify the objects and maps in this case.

Example

Suppose matrices for P, Q, R are stochastic with matrices supported on graphs (so multiple-arrival)

$$Gr(P) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad Gr(Q) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad Gr(R) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Omega_b^P = \{1, 2\}, \quad \mathcal{N}_P(j) = 0, \quad j = 1, 2, 3$$

$$\Omega_b^Q = \{1, 2, 3\}, \quad \mathcal{N}_Q(1) = \mathcal{N}_Q(2) = 0, \quad \mathcal{N}_Q(3) = 1,$$

$$\Omega_b^R = \{1, 2, 3\}, \quad \mathcal{N}_R(1) = \mathcal{N}_R(2) = 0, \quad \mathcal{N}_R(3) = 1,$$

$$C_{\text{env}}^*(\mathcal{T}_+(P)) \not\sim C_{\text{env}}^*(\mathcal{T}_+(Q)) \cong C_{\text{env}}^*(\mathcal{T}_+(R))$$

$$\mathcal{O}_{Gr(P)} \cong \mathcal{O}_{Gr(Q)} \not\sim \mathcal{O}_{Gr(R)}$$

where \cong stands for $*$ -isomorphism and \sim stands for stable isomorphism.