A few unsolved problems

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Problem 1. Let $X = L^{m,p}(\mathbf{R}^n)$. For $E \subset \mathbf{R}^n$, let $X(E)$ be the space of restrictions to E of functions in X.

If $p > n$, it is known that there exists a linear extension opeator $T : X(E) \to X$, with norm at most $C(m, n, p)$.

Also, if $p > n$ and E is finite ($\#(E) = N$), then $\|f\|^p_{X(E)}$ is known to be comparable to a sum of p^{th} powers of $O(N)$ linear functionals $\xi_1, ..., \xi_L$ acting on f. The operator T and the functionals $\xi_1, ..., \xi_L$ can be taken to have a sparse structure, and there are efficient algorithms to compare them. See papers by Fefferman-Israel-Luli on the arXiv.

What if $\frac{n}{m} < p \leq n$? We know almost nothing about this case.

Problem 2. Let X be a metric space and let Y be a Banach space.

For each $x \in X$, let $K(x)$ be a convex subset of Y with dimension at most D.

Suppose that for each $S \subset X$ with at most 2^{D+1} points, there exists a map $F^S : S \to Y$ with Lipschitz constant at most 1, such that

$$F^S(x) \in K(x) \quad \text{for} \quad x \in S.$$

Prove that there exists $F : X \to Y$ with Lipschitz constant at most $C(D)$ such that

$$F(x) \in K(x) \quad \text{for all} \quad x \in X.$$

This question is due to Pavel Shvartsman, motivated by his joint works with Yuri Brudnyi on Whitney's extension problem.

Problem 3. Let X be a finite metric space.

Give an algorithm to decide whether X can be imbedded in a D-dimensional connected Riemannian manifold with sectional curvatures and diameter bounded above by $O(1)$, and with injectivity radius bounded below by 1.

This question comes from Slava Kurylev, Matti Lassas, Sergei Ivanov and Hari Narayanan.

Problem 4. Given a function $f : E \to \mathbf{R}$ decide whether f extends to a C^∞ function.

Problem 5. Fix $m \geq 1$, and let $0 \in E \subset \mathbf{R}^n$.

Let $I(E)$ be the ideal of all m-jets at 0 of functions (in C^m, or maybe in C^∞) vanishing in E. Then $I(E)$ is an ideal in the ring of m-jets at 0.

- Which ideals arise as $I(E)$ for some E?
- Does every $I(E)$ arise already as $I(\hat{E})$ for a semialgebraic set $\hat{E}$?

These questions are due to Nahum Zobin.

Problem 6. Let E be compact.

For each $x \in E$, let $H(x) \subset$ Ring of m-jets at x be a coset of an ideal.
Suppose that

- There exists $F \in C^m(\mathbf{R}^n)$ such that

$$J_x(F) \equiv m\text{-jet of } F \text{ at } x \quad \text{lies in } H(x) \text{ for all } \ x \in E;$$

and

- $\{(x, P) : x \in E, P \in H(x)\}$ is semialgebraic.

Does there exist $F \in C^m(\mathbf{R}^n)$ <u>semialgebraic</u> such that

$$J_x(F) \ \text{ lies in } H(x) \text{ for all } \ x \in E\,?$$

(I'm not sure who first asked this question.)