

# PROJECTION METHODS

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In many mathematical problems motivated by real world applications (Image Reconstruction from Projections, Intensity Modulated Radiation Therapy, to name a few), the goal is to find a point, which will satisfy a given list of restrictions called constraints. When each of the constraints can be described as a closed and convex set, then such a problem is called a *Convex Feasibility Problem* (CFP).

The basic example of the CFP is a system of either linear equations  $Ax = b$  or linear inequalities  $Ax \leq b$ . For simplicity let us have a closer look at the case with the inequalities, which can be written as:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & \leq b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & \leq b_m. \end{cases} \quad (1)$$

Solving system (1) is equivalent to finding a point in the intersection of the half-spaces ( $i$ -th inequality)

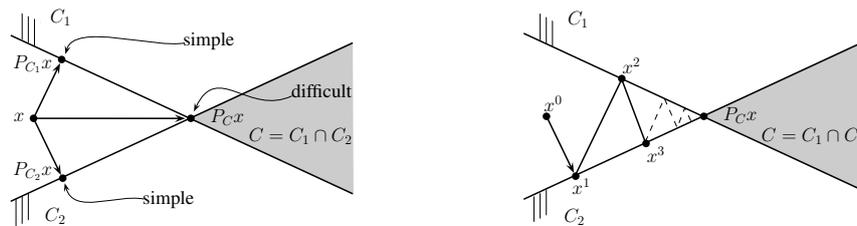
$$C_i := \{x \in \mathbb{R}^n \mid a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i\}, \quad (2)$$

where  $i = 1 \dots, m$ .

In CFP the underlying assumption, notably motivated by applications, is that **one can easily find a point in  $C_i$ , which will minimize the distance between arbitrary chosen  $x$  and  $C_i$** . Such a point, denoted by  $P_{C_i}x$  is called a *metric projection* of  $x$  onto  $C_i$ . For the half-space  $C_i$  defined in (2) the metric projection can even be computed explicitly using the formula

$$P_{C_i}x := x - \frac{\max\{0, \sum_{j=1}^n a_{ij}x_j - b_i\}}{\sum_{j=1}^n a_{ij}^2} \mathbf{a}_i, \quad (3)$$

where  $\mathbf{a}_i$  is the  $i$ -th row of  $A$ . The computation of the metric projection onto  $C_i$  will always guarantee that the resulting point will satisfy at least one of the constraints, namely  $C_i$ . However, this operation won't guarantee that the resulting point will satisfy all the constraints  $C_1, \dots, C_m$ . To overcome this obstacle one could try to compute the metric projection onto the common part of the constraints  $C = \bigcap_{i=1}^m C_i$ , but at this point we reach another underlying assumption, namely, that **the projection onto the intersection of all the constraints is supposed to be difficult**; see Figure 1 (left). Roughly speaking, projection methods are developed for problems, which by their definition satisfy both mentioned assumptions and, in general, they show how to exploit the first property while avoiding the second one. The example of such an approach is illustrated in Figure 1 (right), where the graphical interpretation of the *method of alternating projections* is presented for the system (1) with two inequalities.



**Figure 1** On the left, “simple” projections onto  $C_1$ ,  $C_2$ , and “difficult” projection onto  $C$ . On the right the method of alternating projections

**To summarize**, in this project you will be introduced to a few basic concepts commonly used for the convergence analysis of the classical projection methods. The goal would be to apply some of these techniques to more advanced variants of projection methods and check numerically (using MATLAB) whether such methods are efficient from the computational point of view. Depending on your interests, the project can be either mathematically or computationally oriented.

## Introductory references for projection methods:

- [1] Y. Censor, S. Zenios, *Parallel optimization, Theory, Algorithms and Applications*, Oxford University Press, New York 1997 (Section 5).
- [2] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Springer, Heidelberg, 2012 (Chapter 5, Sections 5.1, 5.4-5.6).

## and a MATLAB text book:

- [3] D. Hanselman, B. Littlefield, *Mastering MATLAB 7*, Pearson Education, Inc., New Jersey 2005.

**Additional Information:** The project will be mentored in English by Rafał Żalas.