Open Problems

1. Charles Fefferman's Problems

Problem 1. Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite. Let $\epsilon > 0$. How many computer operations does it take to compute a function $F_\epsilon \in C^m(\mathbb{R}^n)$ such that $F_\epsilon|_E = f$, with the C^m-norm of F_ϵ at most ϵ percent more than the least value possible (inf)?

Problem 2. Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite. How can we compute a function $F_0 \in C^{m-1,1}(\mathbb{R}^n)$ such that F_0 can be approximated arbitrarily closely in C^{m-1}-norm by functions F_ϵ as in Problem 1, with ϵ arbitrarily small?

Problem 3. Which mathematical theorems are relevant to Problems 1 and 2?
(Think of the Brudnyi-Shvartsman finiteness principle.)

Problem 4. Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$. How can we tell whether there exists $F \in W^{m,p}(\mathbb{R}^n)$ such that $F = f$ on E? If F exists, how small can we take its norm? How can we effectively compute such an F, with its Sobolev norm within a factor C of least possible, if E is finite? Here, C should depend only on m, n, p.

Pavel Shvartsman has answered the above math questions for $m = 1$; likely his ideas will also solve the computer science question in that case.

Problem 5. Let $E \subset \mathbb{R}^n$. How can we decide whether E is a subset of an imbedded (or immersed) compact, C^m-smooth surface of dimension k?
(Think of a Möbius strip in $\mathbb{R}^3$.)

Problem 6. Let $f : E \to \mathbb{R}$, with $E \subset \mathbb{R}^n$ finite. How small can we make

$$\sum_{x \in E} |F(x) - f(x)|^2$$

given that $\|F\|_{C^m(\mathbb{R}^n)}$ has order of magnitude at most M, where M is a given positive number?
(This question is due to **Andrea Bertozzi**).

Problem 7. Suppose we know that $F : \mathbb{R}^n \to \mathbb{R}$ has C^m-norm at most M. Suppose also that we are told the values $F(x_1), \cdots, F(x_N)$ of F at N given points. Our job is to pick additional points $x_{N+1}, \cdots, x_{N+N'}$, and then try to guess F as closely as possible, given the values $F(x_1), \cdots, F(x_{N+N'})$. We pick the points $x_{N+1}, \cdots, x_{N+N'}$ successively, and we are allowed to use $x_1, \cdots, x_{N+k-1}$ and $F(x_1), \cdots F(x_{N+k-1})$ in deciding which point to pick as x_{N+k}. How should we proceed?

The Problem is of course not precisely formulated. Formulate a precise version of the problem and solve it. (This question is due to **Dann Toliver**).

2. Nahum Zobin's Problems

Problem 8. Find an analog of the Whitney Theorem for tensor fields on a manifold with a connection (e.g., a Riemannian manifold).

Problem 9. Extension of functions and tensor fields subject to differential equations and inequalities. In particular, extension of closed differential forms with preservation of closedness.

2.1. **Geometry of open domains and extensions of functions.**

Let Ω be an **open** subset of $\mathbb{R}^n$. Then the notion of a smooth function is well defined, and so are the spaces $C^{m,1}(\Omega)$. We say that $\Omega \in EP(m)$, if

$$C^{m,1}(\mathbb{R}^n)|_{\Omega} = C^{m,1}(\Omega).$$

Whitney proved that if Ω satisfies the Whitney Condition (the geodesic metric in Ω is equivalent to the Euclidean metric, this condition is also called quasi-convexity – see Gromov's book) then $\Omega \in EP(m)$ for all $m \in \mathbb{N}$. One can rather easily show that if $\Omega \in EP(0)$ then Ω is quasi-convex. This means that $\Omega \in EP(0)$ iff Ω is quasi-convex, and the condition $\Omega \in EP(0)$ implies that $\Omega \in EP(m)$ for any $m \in \mathbb{N}$. I have shown that for a finitely connected planar Ω and for any $m \in \mathbb{N}$ the condition $\Omega \in EP(m)$ is equivalent to the condition that Ω is quasi-convex. However, I have also shown that if Ω is an infinitely connected planar domain, or a domain in $\mathbb{R}^n, n \geq 3$, then the condition $\Omega \in EP(m), m \geq 1$, does not imply that $\Omega \in EP(l),\ l \leq m$.

Problem 10. Show that generally (i.e., for an infinitely connected planar domain, or a domain in $\mathbb{R}^n, n \geq 3$) the condition $\Omega \in EP(m)$ does not imply that $\Omega \in EP(m'), 1 \leq m < m'$.

Can one save the implication

$$\Omega \in EP(m),\ m > 0, \Rightarrow \Omega \text{ is quasi-convex}$$

for infinitely connected planar domains Ω by imposing topological restrictions on Ω? Maybe.

Problem 11. Show that if the pairwise distances between components of $\mathbb{R}^2 \setminus \Omega$ are bounded from below, and if $\Omega \in EP(m)$ for some $m \in \mathbb{N}$, then Ω is quasi-convex.

Is there any cure in higher dimensions? Unlikely:

Problem 12. For each $m \in \mathbb{N}$ construct a domain $\Omega_m \subset \mathbb{R}^3$, homeomorphic to an open ball, with boundary smooth at all points, except of one, and such that

$$\Omega \in \left(EP(m) \setminus \bigcup_{0 \leq k < m} EP(k) \right).$$

3. Pavel Shvartsman's problems

The Finiteness Property. We let $C^{k,\omega}(\mathbb{R})$ denote the space of all function $f \in C^k(\mathbb{R})$ whose partial derivatives of order k satisfy the Lipschitz condition (with respect to the metric $\omega(\|x-y\|)$). Recall that this space possesses the following "finiteness property":

There is a positive integer $N = N(k,n)$ such that the following is true: Let f be a function defined on a closed subset $S \subset \mathbb{R}$. Suppose that the restriction $f|_{S'}$ of f to an arbitrary subset $S' \subset S$ consisting of at most N points can be extended to a function $F_{S'} \in C^{k,\omega}(\mathbb{R})$ with norm $\|F_{S'}\|_{C^{k,\omega}(\mathbb{R})} \le 1$. Then the function f itself can be extended to a function $F \in C^{k,\omega}(\mathbb{R})$ with $\|F\|_{C^{k,\omega}(\mathbb{R})} \le \gamma$ where $\gamma = \gamma(n,k)$ is a constant depending only on n and k.

We call the number N appearing in formulations of finiteness properties *"the finiteness number"*. Whitney [25] characterized the restriction of the space $C^k(\mathbb{R}), k \ge 1$, to an arbitrary subset $S \subset \mathbb{R}$ in terms of divided differences of functions. An application of Whitney's method to the space $C^{k,\omega}(\mathbb{R})$ implies the finiteness property for this space with the finiteness number $N(k,1) = k+2$.

Brudnyi and Shvartsman [19, 4] proved that the sharp value of the finiteness number for the space $C^{1,\omega}(\mathbb{R})$ is $N(1,n) = 3 \cdot 2^{n-1}$. Fefferman [6, 8] showed that the finiteness property holds for every $k, n \ge 1$. An upper bound for the finiteness number $N(k,n)$ given in [6, 8] is

$$N(k,n) \le (\dim \mathcal{P}_k + 1)^{3 \cdot 2^{\dim \mathcal{P}_k}}.$$

Here $\mathcal{P}_k$ stands for the space of polynomials of degree at most k defined on $\mathbb{R}$.

Basing on this estimate of $N(k,n)$ Bierstone and Milman [1] and Shvartsman [23] proved that the finiteness property for $C^{k,\omega}(\mathbb{R})$ holds with the finiteness number $N(k,n) = 2^{\dim \mathcal{P}_k}$.

Problem 13. *Find the sharp value of the "finiteness number" $N = N(k,n)$ for the space $C^{k,\omega}(\mathbb{R}^n)$ for $k > 1$.*

In [23] we conjectured the following:

Conjecture 13.1. *The sharp value of the finiteness number for $C^{k,\omega}(\mathbb{R}^n)$ equals*

$$N(k,n) = \prod_{m=0}^{k} (k-m+2)^{\binom{n+m-2}{m}}. \tag{1}$$

In particular, in the two dimensional case, (1) is the same as $N(k,2) = (k+2)!$ and so the first step towards resolving this conjecture might be to consider the simplest case of the space $C^{2,\omega}(\mathbb{R}^2)$ and to ask:

Is it true that in this case $N(2,2) = 24$?

2. Trace criterion for the space $C^{k,\omega}(\mathbb{R}^n)$. As we have noted above, Whitney [25] characterized the restriction $C^k(\mathbb{R})|_S$ in terms of divided differences of functions. A similar intrinsic characterization of the trace space $C^{k,\omega}(\mathbb{R})|_S$ has been obtained by Merrien [17].

Recall a trace criterion for the space $C^{1,\omega}(\mathbb{R}^2)\mid_S$, $S\subset\mathbb{R}^2$, presented in the author's paper [21]. (Note that by the finiteness theorem this criterion is expressed in terms of exactly 6 (arbitrary!) points of S.)

Let $Z\subset S$ be an arbitrary set consisting of three points and let θ_Z be the biggest angle of the triangle whose vertices are the points of Z. We let P_Z denote the affine polynomial interpolating f at the points of Z.

A locally bounded function f is in $C^{1,\omega}(\mathbb{R}^2)\mid_S$ if and only if there exists $\lambda>0$ such that the following inequalities hold:

(i). for every subset $Z=\{z_0,z_1,z_2\}\subset X$ such that z_1 belongs to the line segment $[z_0,z_2]$

$$\left|\frac{f(z_0)-f(z_1)}{\|z_0-z_1\|}-\frac{f(z_1)-f(z_2)}{\|z_1-z_2\|}\right|\le\lambda\omega(\|z_0-z_2\|)\ ;$$

(ii). for every pair of subsets $Z_1,Z_2\subset S$ each consisting of three non-collinear points

$$\|\nabla P_{Z_1}-\nabla P_{Z_2}\|\ \le\ \lambda\left\{\frac{\omega(\operatorname{diam}Z_1)}{\sin\theta_{Z_1}}+\frac{\omega(\operatorname{diam}Z_2)}{\sin\theta_{Z_2}}+\omega\left(\operatorname{diam}Z_1\cup Z_2\right)\right\}.$$

Moreover, $\|f\|_{C^{1,\omega}(\mathbb{R}^2)|_S}\sim\inf\lambda$.

Observe that the proofs of the finiteness property for the space $C^{k,\omega}(\mathbb{R}^n)$ given in [19, 4] ($k=1$) and [6, 8] (arbitrary $k,n\ge1$) are constructive. However, straightforward application of these algorithms to $N(k,n)$-element sets leads to very complicated trace criterions.

Problem 14. *Find a trace criterion for the space $C^{k,\omega}(\mathbb{R}^n)$ for $n>1$.*

Thus, the first step towards resolving this problem might be to consider the simplest case of the space $C^{1,\omega}(\mathbb{R}^3)$. Observe that, by the "finiteness property" for this space, the corresponding criterion should be expressed in terms of exactly 12 (arbitrary!) points of $S\subset\mathbb{R}^3$.

3. The Whitney extension problem for the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$. Let m be a non-negative integer. We let Ω_m denote the class of non-decreasing continuous functions $\omega:\mathbb{R}_+\to\mathbb{R}_+$ such that $\omega(0)=0$ and the function $\omega(t)/t^m$ is non-increasing. Given non-negative integers k and m and $\omega\in\Omega_m$ we define the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$ as follows: a function $f\in C^k(\mathbb{R}^n)$ belongs to $C^k\Lambda^m_\omega(\mathbb{R}^n)$ if there exists $\lambda>0$ such that for every multi-index α, $|\alpha|=k$, and every $x,h\in\mathbb{R}^n$ we have $|\Delta^m_h(D^\alpha f)(x)|\le\lambda\omega(\|h\|)$.

Here as usual $\Delta^m_h f$ denotes the m-th difference of a function f of step h. $C^k\Lambda^m_\omega(\mathbb{R}^n)$ is normed by

$$\|f\|_{C^k\Lambda^m_\omega(\mathbb{R}^n)}:=\sum_{|\alpha|\le k}\sup_{x\in\mathbb{R}^n}|D^\alpha f(x)|+\sum_{|\alpha|=k}\sup_{x,h\in\mathbb{R}^n}\frac{|\Delta^m_h(D^\alpha f)(x)|}{\omega(\|h\|)}.$$

In particular, for $m=1$ and $\omega\in\Omega_1$ the space $C^k\Lambda^1_\omega(\mathbb{R}^n)$ coincides with the space $C^{k,\omega}(\mathbb{R}^n)$. In turn, the space $\Lambda^m_\omega(\mathbb{R}^n):=C^0\Lambda^m_\omega(\mathbb{R}^n)$, $\omega\in\Omega_m$, coincides with the generalized Zygmund space of bounded functions f on $\mathbb{R}^n$ whose modulus of smoothness of order m, $\omega_m(t;f)$, satisfies the inequality $\omega_m(t;f)\le\lambda\omega(t)$, $t\ge0$. In particular, the space

$\Lambda^2_\omega(\mathbb{R}^n)$ with $\omega(t) = t$ is the classical Zygmund space of bounded functions satisfying the Zygmund condition: there is $\lambda > 0$ such that for all $x, y \in \mathbb{R}^n$

$$|f(x) - 2f\left(\tfrac{x+y}{2}\right) + f(y)| \le \lambda\|x - y\|.$$

Consider three Whitney's type problems for the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$.

Problem 15. *Whether the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$ possesses the finiteness property?*

For $n = 1$ the finiteness property follows from the results of Merrien [17], Jonsson [16], Shevchuk [18] and Galan [12]. If $n > 1$, the answer is positive for $m = 1$ (see Section 1), and for $k = 0, m = 2$, i.e., for the Zygmund space, see [19] (in this case the optimal finiteness number $N = 3 \cdot 2^{n-1}$ is the same as for $C^{1,\omega}(\mathbb{R}^n)$.)

Problem 16. *Given closed subset $S \subset \mathbb{R}^n$, does there exist a linear continuous extension operator $T : C^k\Lambda^m_\omega(\mathbb{R}^n)|_S \to C^k\Lambda^m_\omega(\mathbb{R}^n)$?*

The answer is positive for $m = 1$ and arbitrary $n > 1$ (for $k = 1$ see Brudnyi and Shvartsman [3], for $k > 1$ see Fefferman [7, 10]); for $k = 0, m = 2$ see [3].

Problem 17. *Find a trace criterion for the space $C^k\Lambda^m_\omega(\mathbb{R}^n)(\mathbb{R}^n)$.*

We have such a criterion only for the space $C^{1,\omega}(\mathbb{R}^2)$ (see Section 2) and $\Lambda^2_\omega(\mathbb{R}^2)$ [21].

4. Sobolev Extension Domains. Given positive integer k and $p \ge 1$, a domain Ω in $\mathbb{R}^n$ is said to be Sobolev W^k_p-extension domain if the following isomorphism

$$W^k_p(\Omega) = W^k_p(\mathbb{R}^n)|_\Omega$$

holds. In other words, Ω is a Sobolev extension domain if every Sobolev function $f \in W^k_p(\Omega)$ can be extended to a Sobolev W^k_p-function F defined on all of $\mathbb{R}^n$.

A domain with locally Lipschitz boundary provides an example of a Sobolev extension domain for every $k, p \ge 1$. The same is true for the class (ε, δ)-domains introduced by Jones [15].

Also recall that the case $p = \infty$ has been studied by Whitney [26] who proved that quasi-Euclidean domains are W^k_∞-extension domains for every $k \ge 1$ (Ω is quasi-Euclidean if its inner (or geodesic) metric is equivalent to the Euclidean distance). Zobin [28] showed that every finitely connected bounded planar W^k_∞-extension domain is quasi-Euclidean. He also showed, see [27], that for every $k \ge 2$ there exists a bounded planar W^k_∞-extension domain which is not quasi-Euclidean.

Goldshtein, Latfullin and Vodopyanov [13, 14] proved that every simply connected bounded planar domain is a W^1_2-extension domain if and only if Ω is a quasi-disk, i.e., the image of a disk under a quasi-conformal mapping of the plane onto itself.

Problem 18. *Given $p \ge 1, n > 1, k \ge 1$ find a geometric description of the class of Sobolev W^k_p-extension domains in $\mathbb{R}^n$.*

Thus, the first step towards resolving this problem might be to consider the simplest case of a simply connected planar W^1_p-extension domain with $p \neq 2, \infty$.

5. A geometrical background of the finiteness property: some connections with convex geometry. The proof of the finiteness property for the space $C^{1,\omega}(\mathbb{R}^n)$ presented in [19, 4] is based on the following geometrical result expressed in terms of *set-valued maps* and their *Lipschitz selections*.

Let $(\mathcal{M}, d)$ be a metric space and let F be a set-valued mapping from $\mathcal{M}$ into the family $Conv(\mathbb{R}^n)$ of all convex compact subsets of $\mathbb{R}^n$.

Problem 19. *Find conditions on F for which it has a Lipschitz selection.*

Recall that a mapping $f : \mathcal{M} \to \mathbb{R}^n$ is said to be a Lipschitz selection of F if $f(x) \in F(x)$ for every $x \in \mathcal{M}$ and $f \in \mathrm{Lip}(\mathcal{M}, \mathbb{R}^n)$.

Conjecture 19.2. *(See [2]). Let F be a set-valued mapping from a metric space $(\mathcal{M}, d)$ into $Conv(\mathbb{R}^n)$. Suppose that, for every subset $\mathcal{M}' \subset \mathcal{M}$ consisting of at most 2^n points, the restriction $F\mid_{\mathcal{M}'}$ of F to $\mathcal{M}'$ has a Lipschitz selection $f_{\mathcal{M}'}$ such that $\|f_{\mathcal{M}'}\|_{\mathrm{Lip}(\mathcal{M}',\mathbb{R}^n)} \le 1$. Then F has a Lipschitz selection f with $\|f\|_{\mathrm{Lip}(\mathcal{M},\mathbb{R}^n)}$ bounded by some constant $\gamma = \gamma(n)$ depending only on n.*

Note that in the case of the trivial pseudometric $d \equiv 0$ the conjecture is true, even with $n+1$ instead of 2^n, since in this case it is exactly the classical theorem of Helly [5]. In [20, 21, 22] we proved the following:

(1). The conjecture is true for $\mathbb{R}^2$.
(2). The conjecture is true for every finite k-point metric space $\mathcal{M}$, but with $\gamma = \gamma(n, k)$.
(3). If the conjecture is true for some number $N(n)$ in place of 2^n then it also holds in its original form.
(4). The conjecture is false in general if 2^n is replaced by some number $N(n)$ with $N(n) < 2^n$.
(5). The conjecture is true for set-valued maps F which take values in the class $\mathcal{A}(\mathbb{R}^n)$ of all affine subsets of $\mathbb{R}^n$. (In case of the Whitney extension problem for $C^{1,\omega}(\mathbb{R}^n)$ we only need to consider this class of convex subsets of $\mathbb{R}^n$.)

Thus the first step towards resolving this conjecture might be to consider the simplest case of the metric space $\mathcal{M} = \{1, 2, ..., m\}$ and $n = 3$.

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4. Edward Bierstone and Pierre Milman's Problems

Extension problems for geometric classes (E.g., subanalytic, semialgebraic or, more generally, o-minimal structures).

Problem 20. Extension of subanalytic functions Is there a Whitney extension theorem for C^m subanalytic functions on a closed subanalytic subset X of $\mathbb{R}^n$? (A function is *subanalytic* if its graph is subanalytic.)

As evidence, there seems to be a subanalytic extension involving loss of differentiability, as in our paper [Inv. Math. 151 (2003), 329–352].

2. Characterization of "tame" subanalytic sets by the extension property There is a remarkable subclass of subanalytic sets, called *semicoherent*, characterized by the following theorem.

Theorem 4.1. *Let X denote a compact subanalytic subset of $\mathbb{R}^n$. Then the following conditions are equivalent:*

(1) *Composite function property. If $p : M \to \mathbb{R}^n$ is a proper real-analytic mapping with $p(M) = X$, then the ring of composite C^∞ functions $p^*C^\infty(\mathbb{R}^n)$ (where $p^*(g) := g \circ p$) is closed in $C^\infty(M)$.*

(2) *$C^\infty(X)$ is the intersection of all $C^m(X)$.*

(3) *Natural local algebraic invariants of X (e.g., the Hilbert-Samuel function) are upper-semicontinuous (in the subanalytic Zariski topology).*

(4) *For each l, the degree $\leq l$ part of the C^m paratangent bundle ($m \geq l$) stabilizes as m increases.*

(5) *X is semicoherent (i.e., satisfies a stratified version of the Oka-Cartan coherence theorem).*

(6) *There is a uniform bound for a local invariant of X called the Chevalley function (which compares algebraic and metric notions of order of vanishing).*

The various equivalences are proved in [Ann. of Math. 147 (1998), 731–785], [Duke Math. J. 83 (1996), 607–620], [Inv. Math., loc. cit.]. We show that, if X is semicoherent, then there is an extension operator $E : C^\infty(X) \to C^\infty(R^n)$. Thus we get estimates on the C^k seminorms of an extension: Given $k \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ compact, there exist $l = l(k, K) \in \mathbb{N}$ and $L = L(k, K) \subset X$ compact, such that

$$\|E(f)\|_k^K \leq c\|f\|_l^L.$$

Problem 21. Can the class of semicoherent sets be characterized by the extension property?

We prove:

Theorem 4.2. *Suppose there is an extension operator as above with an estimate $l(0, K) = 0$ on the zeroth seminorms, for every compact K. Then X is semicoherent.*

So the preceding question can be reformulated: *Does semicoherence imply the extension property with $l(0, K) = 0$?*

We do not know whether this is true is simple examples (e.g., $X =$ union of the x-axis and the parabola $y = x^2$ in $\mathbb{R}^2$).