Cycles in Random Permutations

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Let S_n denote the set of permutations of $[n] =: \{1, \cdots, n\}$. The so-called "Feller construction" gives a very natural way to construct a uniformly random permutation from S_n—by uniformly random we mean that for any $\sigma \in S_n$, the probability that the randomly constructed permutation is equal to σ is $\frac{1}{n!}$. In the text below, whenever we mention selecting a "random" element from a finite set, we mean uniformly random, that is, equal probabilities for choosing any particular element. The Feller construction will be presented in our introductory lecture and we will show that it is equivalent to construction presented in the following paragraph. Whereas it is immediately obvious that the Feller construction generates a uniformly random permutation, it is not immediately obvious that the other construction also does.

Here is the other construction: Let $A_0 = [n]$ and let $N_0 = |A_0|$ $(= n)$. Choose a number at random from $[N_0]$; call it M_1. Now choose randomly M_1 elements from A_0. Denote this set by B_1 and order these M_1 elements randomly into an M_1-cycle. This finishes the first stage. Now let $A_1 = A_0 - B_1$ and $N_1 = N_0 - M_1$. Choose a number at random from $[N_1]$; call it M_2. Choose randomly M_2 elements from A_1. Denote this set by B_2 and order these M_2 elements randomly into an M_2-cycle. The process continues like this for k steps, where k is the first positive integer for which $N_k = M_k$. We demonstrate the construction with an example: Let $n = 8$. So $A_0 = [8]$ and $N_0 = 8$. Let's say we randomly select $M_1 = 5$ from $[N_0]$ and $B_1 = \{2, 3, 5, 7, 8\}$ from A_0. And let's say we randomly build from B_1 the 5-cycle $(3, 2, 8, 5, 7)$. Now we have $A_1 = A_0 - B_1 = \{1, 4, 6\}$ and $N_1 = N_0 - M_1 = 3$. Say we now randomly pick $M_2 = 1$ from $[N_1]$ and $B_2 = \{4\}$ from A_1. We have no choice now but to build the 1-cycle (4). Now we have $A_2 = A_1 - B_2 = \{1, 6\}$ and $N_2 = N_1 - M_2 = 2$. Say we now pick $M_3 = 2$ from $[N_2]$. Then we have no choice but to let $B_3 = \{1, 6\} = A_2$ and to build the 2-cycle $(1, 6)$. The process now stops and we have constructed the permutation whose cycle representation is $(3, 2, 8, 5, 7)(4)(1, 6)$.

We now note several important facts about cycles in uniformly random permutations. Let $C_j^{(n)}$ denote the number of cycles of length j in a uniformly random permutation from S_n. Then $C_j^{(n)}$ is a random variable. A classic result in the theory of random permutations states that $C_j^{(n)}$ converges *weakly* as $n \to \infty$ to the Poisson distribution with parameter $\frac{1}{j}$, which is equivalent to saying that $\lim_{n\to\infty} P(C_j^{(n)} = k) = e^{-\frac{1}{j}} \frac{(\frac{1}{j})^k}{k!}$, $k = 0, 1, \cdots$. Also the expected value $EC_j^{n)}$ of $C_j^{(n)}$ satisfies $\lim_{n\to\infty} EC_j^{(n)} = \frac{1}{j}$, which is the expected value of a random variable with the Pois$(\frac{1}{j})$ distribution. Furthermore, if we let N_n denote the total number of cycles in the random permutation, then the expected value EN_n of N_n satisfies $EN_n \sim \log n$ as $n \to \infty$ and the weak law of large numbers holds: for any $\epsilon > 0$, $\lim_{n\to\infty} P(|\frac{N_n}{EN_n} - 1| > \epsilon) = 0$.

In this project, we will investigate the behavior of cycles under a different probability measure on S_n, one which is obtained via a construction very similar to the one noted above. The only difference is this: at each step j, when we choose the set B_j with M_j elements, *instead of ordering them randomly into an M_j-cycle, we order them randomly into a permutation.* That is, for example, above we had $B_1 = \{2, 3, 5, 7, 8\}$ from which we had to choose randomly one of the 4! possible 4-cycles of these elements, and we randomly chose the cycle $(3, 2, 8, 5, 7)$. Now, instead we randomly choose one of the 5! *permutations* of the elements $\{2, 3, 5, 7, 8\}$. Since this permutation will have at least one cycle and maybe more, it is clear that typical permutations under this new measure will have more cycles than will typical permutations under the uniform measure. Thus, we expect $EC_j^{(n)}$ and EN_n under this new measure to be larger than they were under the uniform measure. In this project, we investigate this point. In addition, we would like to investigate the probability of an *inversion* under this new measure. A pair i, j with $i < j$ and $i, j \in S_n$ is called an inversion for a permutation $\sigma \in S_n$ if $\sigma_i > \sigma_j$. Under the uniform measure the probability that the pair i, j forms an inversion is $\frac{1}{2}$; what is it under this new measure?