PROBLEMS POSED AT OPEN PROBLEMS SESSION IN NAHSHOLIM - MAY 29 2018

The problem session was moderated by Mladen Bestvina. Names attached to questions are the people who posed the questions at the conference, and may not be the originator of the problem.

Question 1 (Wise)**.** *Let G be a hyperbolic group which is a subgroup of a right-angled Artin group. Is G the fundamental group of a compact non-positively curved cube complex?*

Question 2 (Bestvina)**.** *Suppose that G is hyperbolic. Does G act on a quasi-tree so that every infinite order element acts loxodromically?*

Question 3 (Futer)**.** *Under the same hypotheses as Question 1, if G is 1–ended does G contain a quasi-convex surface subgroup?*

Question 4 (Chatterji)**.** *If a group G acts properly on some quasi-tree, does G act properly on some ℓ^p–space?*

Question 5 (Wise)**.** *Suppose G is locally indicable and locally quasi-convex. Is G the fundamental group of a compact non-positively curved cube complex? What about if we also assume G is hyperbolic?*

Question 6 (Lazarovich)**.** *Let G be hyperbolic and suppose that $g \in G$ has infinite order. It need not be the case that there is a surface subgroup in G which contains g. An example is given by gluing a surface with boundary to a circle via a 3–to–1 map from the boundary to the circle. The circle is not contained in any surface subgroup.*

Is it possible that every g of infinite order satisfies $\langle g\rangle \cap S \neq \{1\}$ for some surface subgroup S?

Question 7 (Lazarovich)**.** *Suppose that H is hyperbolic. Does there exist G hyperbolic with $H \leq G$ so that for every $g \in H$, g lies in a surface subgroup of G?*

Question 8 (Ruane)**.** *Suppose G acts geometrically on $CAT(0)$ spaces X and Y. Does the G–equivariant quasi-isometry between X and Y induce a bijection on the connected components of the Tits boundaries?*

Question 9 (Ruane)**.** *If you change the Croke–Kleiner example by turning the middle square torus into a rectangular one, does the boundary change (up to homeomorphism)?*

Date: June 10, 2018.

Question 10 (Ruane). *Does there exist a "non-cubical Croke–Kleiner example"?*

Dani Wise suggested looking at tubular groups, as in his paper "Cubular tubular groups".

Question 11 (Swenson). *Suppose that X is a CAT(0) space and that G acts on X geometrically. When does ∂X determine $\partial_T X$?*

It does when

(1) ∂X is totally disconnected;
(2) $\partial X = \Sigma C$ is a suspension of the Cantor set C (Ruane);
(3) $\partial X = C * C$ or $\partial X = \Sigma(C * C)$ (Chao–Swenson).

It does not when ∂X is a sphere.

What about $C * C * C$?

Question 12 (Stark). *Suppose that X and Y are CAT(0) spaces and that G acts geometrically on both X and Y. If ∂X is planar, is ∂Y planar?*

There are examples where ∂X contains non-planar graphs, but ∂Y does not. However, in the known examples ∂Y (which is not locally connected) is non-planar. Swenson suggested looking at other characterizations of planarity.

Question 13 (Mahan). *Suppose $\Gamma < \mathrm{SO}(n,1)$ is a uniform lattice. Is Γ cubulated?*

Futer notes that this is open even if Γ contains an embedded totally geodesic hypersurface.

Conjecture 1 (Wise). Γ *as in Question 13 (with $n > 3$). Then* Γ virtually algebraically fibers *in the sense that there is a finite-index subgroup $\Gamma' < \Gamma$, and a finitely generated normal $N \trianglelefteq \Gamma'$ with a short exact sequence:*

$$1 \to N \to \Gamma' \to \mathbb{Z} \to 1.$$

There are only two known such examples.

Question 14 (Walsh). *Suppose that $(G, \mathscr{P})$ is relatively hyperbolic and that the Bowditch boundary $\partial(G, \mathscr{P})$ is planar, connected, nonempty and does not contain a cut point. Is G virtually Kleinian?*

The note-taker remarks that (since S^2 is planar) this conjecture includes the Cannon Conjecture.

Question 15 (Woodhouse). *Suppose that X is CAT(0) and that G acts geometrically on X. Suppose that X contains a flat (an isometrically embedded copy of $\mathbb{E}^2$). Does G contain $\mathbb{Z}^2$?*

[The expected answer is that this does not always hold.]

Question 16 (Mahan). *Can a closed hyperbolic $(2n+1)$–manifold fiber over the circle with $n > 1$?*

Question 17 (Hruska). *In the context of Question15, if G does not have isolated flats, does G contain $F_2 \times \mathbb{Z}$?*

[Wise asks, what about if X is a product of two trees?]

Question 18 (Wise). *Suppose that G is a finitely generated group and that Γ is a Cayley graph of G. Let $H < G$, and define the* growth of H in Γ *to be*

$$\lambda_H = \limsup_{n\to\infty} \sqrt[n]{B_n \cap H},$$

where B_n is the ball of radius n about 1 in Γ.

Say that G has (quasi-convex) growth gap *if there exists $0 < \varepsilon < \lambda_G$ so that whenever H is a (quasi-convex) subgroup which has infinite index in G we have $\lambda_H \leq \varepsilon$.*

Note that Dahmani–Futer-Wise prove that F_2 has no growth gap. Li–Wise prove that if G is special compact then there is no growth gap.

On the other hand, Coulon–Dal'Bo–Sambusetti proved that if G has Property (T) and is hyperbolic then G has a growth gap.

- *Does having a growth gap depend on the generating set of G?*
- *Does B_4, the braid group on 4 strands, have growth gap?*