Dessin d'Enfant: Drawing on Surfaces

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Dessin d'enfant are deeply related to questions in arithmetic, algebraic geometry, complex analysis and more. By studying simple "child's drawings" we will be able to solve complicated problems in these fields. A *dessin d'enfant* ("child's drawing") is a connected graph in which each vertex is colored black and white, and the two ends of every edge have different colors. Dessin d'enfant were first promoted in 1984 by Alexandre Grothendieck, one of the fathers of algebraic geometry. Below are a few examples.

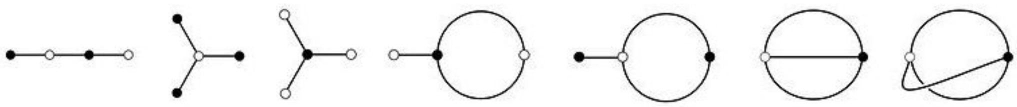
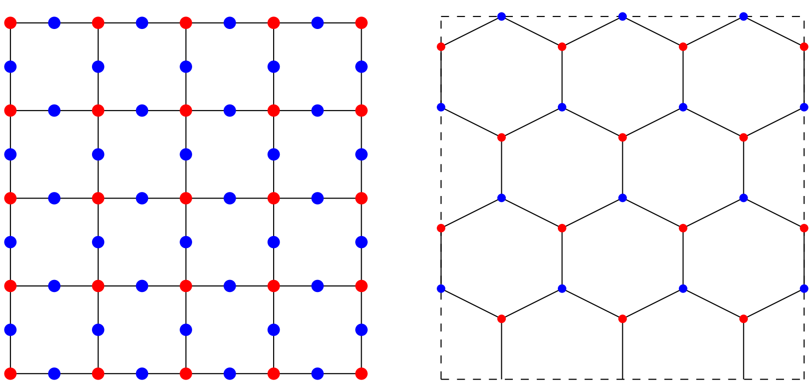


Figure 1: Dessin of degree 3 (nonintersecting on a sphere)

Dessins arise from maps from a surface to a Riemann sphere, where the map is a local homeomorphism outside the points $0, 1, \infty$. (Pre-images of a point neighborhood look like a stack of pancakes.) Given such a map, we can draw a dessin with black vertices that are inverse images of 0, white vertices that are inverse images of 1, and the edges of the dessin are the components of the inverse image of the line segment $(0, 1)$. Conversely, given a dessin, there is a way to construct a map from a surface to the Riemann sphere. Proving nonexistence of some dessin is a way to prove nonexistence of certain maps.

In this project, we consider dessin d'enfant on surfaces other than a sphere. For example, there are four families of regular dessin d'enfant on a torus, which correspond to regular tilings. Two are drawn in figure 2. We use the usual gluing construction to make a torus from a square: glue the top and bottom, and the right and left.



(a) [2*][4*][4*]   (b) [3*][3*][3*]

Figure 2: Two regular dessin on a torus

In our notation, the last number corresponds to the degree of each face, and the first numbers are the degree of each vertex. So for example, in the [3*][3*][3*] tiling, each vertex has degree 3, and each face has degree 3.

We ask what happens if we change the type of the tiling slightly from the regular type, for example to $[3^*][3^*][2, 4, 3^*]$. In this tiling every vertex has degree 3, and every face except two of the faces has degree 3. Of the 2 different faces, one has degree 2 and the other degree 4. We can show it is impossible to draw such a tiling.

In this project, we will look at tilings on surfaces of genus larger than 2, we can make these tilings by folding up a tiling of the hyperbolic plane. We conjecture that for such hyperbolic tilings, all small changes to a dessin can be realized. We will try to prove this.

Prerequisites: no knowledge of algebraic geometry or complex analysis is needed. Some understanding of topology will be useful, but is not necessary. If you like geometry, algebra, topology, combinatorics and discrete math, this is a good project for you. Programming skills are also an asset.