Distances between homotopy classes of self maps of the unit circle

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Consider two smooth maps $f, g$ from the unit circle $S^1$ to itself. A natural question is

**Question:** Is there a universal $\delta > 0$ such that the inequality $\|f - g\| < \delta$ implies that $f$ and $g$ have the same degree?

The degree of a continuous map $f : S^1 \to S^1$ can be defined as follows. Write $f(e^{i\theta}) = e^{i\varphi(\theta)}$ with $\varphi : [0, 2\pi] \to \mathbb{R}$. Since we must have $e^{i\varphi(2\pi)} = e^{i\varphi(0)}$ it follows that $\varphi(2\pi) - \varphi(0) = 2\pi d$ for some $d \in \mathbb{Z}$. We then define $\deg f = d$.

The answer to the question depends of course on the norm $\| \cdot \|$. When one uses the maximum norm $\|h\|_\infty = \max_{x \in S^1} |h(x)|$, the answer to the question is “yes” and the optimal $\delta$ equals 2. Indeed, if $\|f - g\|_\infty < 2$ then there is no point $x \in S^1$ s.t. $f(x) = -g(x)$. This implies that the value $-1$ is not in the image of the map $h = f/g = \bar{f}g$. Therefore, we must have $\deg h = 0$, implying that $\deg f = \deg g$.

It is interesting to investigate the above question for other norms, or seminorms. The special case of the $W^{1,p}$-seminorm, $\|h\|_{W^{1,p}} = \left( \int_{S^1} |h'|^p \right)^{1/p}$, for any $p \in [1, \infty)$ was studied in [1]. From the results in [1] one gets that the answer to the question is “yes” with optimal $\delta = 2^{(1/p)+1} \pi^{(1/p)-1}$. This is a direct consequence of the formula proved in [1] for the $W^{1,p}$-distance between homotopy classes:

$$\text{dist}_{W^{1,p}}(E_{d_1}, E_{d_2}) := \inf_{f \in E_{d_1}} \inf_{g \in E_{d_2}} d_{W^{1,p}}(f, g) = 2^{(1/p)+1} \pi^{(1/p)-1} |d_1 - d_2|,$$

where $E_d := \{ f \in C^\infty(S^1; S^1) : \deg f = d \}$.

Some estimates for the distances between homotopy classes for other $W^{m,p}$-seminorms were obtained in [2]. However no explicit formulas analogous to (1) are known even when $m = 2$. Obtaining such formulas is one of the main objectives of the proposed project.

**Pre-requisites.** The minimal required knowledge is Calculus at the level of first year. Any knowledge in Real Analysis and/or Functional Analysis could be useful, but is not obligatory.
References
