Distances between homotopy classes of self maps of the unit circle

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Consider two smooth maps f,g from the unit circle $\mathbb{S}^1$ to itself. A natural question is

Question: Is there a universal $\delta>0$ such that the inequality $\|f-g\|<\delta$ implies that f and g have the same degree?

The *degree* of a continuous map $f:\mathbb{S}^1\to\mathbb{S}^1$ can be defined as follows. Write $f(e^{i\theta})=e^{i\varphi(\theta)}$ with $\varphi:[0,2\pi]\to\mathbb{R}$. Since we must have $e^{i\varphi(2\pi)}=e^{i\varphi(0)}$ it follows that $\varphi(2\pi)-\varphi(0)=2\pi d$ for some $d\in\mathbb{Z}$. We then define $\deg f:=d$.

The answer to the question depends of course on the norm $\|\cdot\|$. When one uses the *maximum norm* $\|h\|_\infty=\max_{x\in\mathbb{S}^1}|h(x)|$, the answer to the question is "yes" and the optimal δ equals 2. Indeed, if $\|f-g\|_\infty<2$ then there is no point $x\in\mathbb{S}^1$ s.t. $f(x)=-g(x)$. This implies that the value -1 is not in the image of the map $h=f/g=f\bar{g}$. Therefore, we must have $\deg h=0$, implying that $\deg f=\deg g$.

It is interesting to investigate the above question for other norms, or seminorms. The special case of the $W^{1,p}$-seminorm, $\|h\|_{W^{1,p}}=\left(\int_{\mathbb{S}^1}|h'|^p\right)^{1/p}$, for any $p\in[1,\infty)$ was studied in [1]. From the results in [1] one gets that the answer to the question is "yes" with optimal $\delta=2^{(1/p)+1}\pi^{(1/p)-1}$. This is a direct consequence of the formula proved in [1] for the $W^{1,p}$-distance between homotopy classes:

$$\mathrm{dist}_{W^{1,p}}(\mathscr{E}_{d_1},\mathscr{E}_{d_2}):=\inf_{f\in\mathscr{E}_{d_1}}\inf_{g\in\mathscr{E}_{d_2}}d_{W^{1,p}}(f,g)=2^{(1/p)+1}\pi^{(1/p)-1}|d_1-d_2|,\tag{1}$$

where $\mathscr{E}_d:=\{f\in C^\infty(\mathbb{S}^1;\mathbb{S}^1):\deg f=d\}$.

Some estimates for the distances between homotopy classes for other $W^{m,p}$-seminorms were obtained in [2]. However no explicit formulas analogous to (1) are known even when $m=2$. Obtaining such formulas is one of the main objectives of the proposed project.

Pre-requisites. The minimal required knowledge is Calculus at the level of first year. Any knowledge in Real Analysis and/or Functional Analysis could be useful, but is not obligatory.

References

[1] J. Rubinstein and I. Shafrir, *The distance between homotopy classes of $\mathbb{S}^1$-valued maps in multiply connected domains*, Israel J. Math., **160** (2007), 41–59.

[2] H. Brezis, P. Mironescu and I. Shafrir, *Distances between homotopy classes of* $W^{s,p}(\mathbb{S}^N;\mathbb{S}^N)$, ESAIM Control Optim. Calc. Var. **22** (2016), 1204–1235.