HOW MANY WAVE LENGTHS CAN FIT IN A QUANTUM GRAPH

LIOR ALON, RAM BAND

Abstract

In this project we will face some open questions in *spectral geometry of metric graphs* (A subject also known as Quantum Graphs), which may have impact on the quantum graphs community. This type of research is a symbiosis of numeric work (in which we gather evidence in order to produce conjectures) with analysis (where we prove our conjectures). The following section is a brief introduction in order to introduce the subject of research and provide some motivation. The main goals and requirements are presented after the introduction.

Introduction

Consider the solutions of the following ODE on the interval $I = [0, L]$:

$$-f'' = k^2 f$$
$$f'(0) = f'(L) = 0.$$

The set of eigenvalues, $\{k_n^2\}_{n=0}^{\infty}$ is called spectrum. The solutions, namely the eigenfunctions $\{f_n\}_{n=0}^{\infty}$ and their eigenvalues can be considered as the harmonics of a guitar string, or standing waves if we consider the wave equation. These solutions are also wave functions of the states of a free one dimensional particle according to quantum mechanics. These solutions can be written explicitly in this specific case

$$f_n(x) = \cos(k_n x)$$
$$k_n = \frac{\pi n}{L}.$$

If we consider more complicated domains, such as graphs instead of a single segment, then such solutions will not have an explicit form. We therefore look for quantities which characterize these solutions. An important such quantity is the wave length $\lambda_n = \frac{2\pi}{k_n}$ (see https://en.wikipedia.org/wiki/Wavelength, for physical motivation and nice pictures). It is not hard to see that the ratio $\frac{Lk_n}{\pi} = 2\frac{L}{\lambda_n} = n$, which is twice "the number of wave lengths that fit in the segment" provides the spectral position (namely n) in this case (try to count wave lengths in Fig (0.1)).

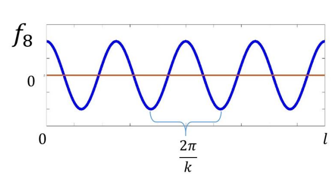

Figure 0.1. The 8th eigenfunction and its corresponding wave length.

Now consider a more general case, a collection of segments $\{I_j\}_{j=1}^{E}$of lengths $\{L_j\}_{j=1}^{E}$ connected by vertices (This is called a metric graph, see Fig (0.2)(i) for example). We can talk about common solutions to

$$-f'' = k^2 f,$$

for all edges, with appropriate vertex conditions (that is boundary conditions for each edge). This system is called a **Quantum Graph**, and in this case the solutions can not be written explicitly. It is an interesting fact the ratio $2\frac{L}{\lambda_n}$ (where $L = \sum_{j=1}^{E} L_j$ is the length of the entire metric graph) will go asymptotically like $'n'$, but in general $2\frac{L}{\lambda_n}$ is **not even integer valued**!

As before, we can interpret $2\frac{L}{\lambda_n}$ as twice *"the number of wave lengths that fit in the quantum graph"* ,(In order to understand the complexity try to count wave lengths in Fig (0.2)(ii)). It may become clear now that in order to answer *"how many wave lengths can fit in a quantum graph",* we need to count how many wave lengths fit in each edge (that is $2\frac{L_j}{\lambda_n}$), and in particular we are interested in the **fractional part of these numbers!**

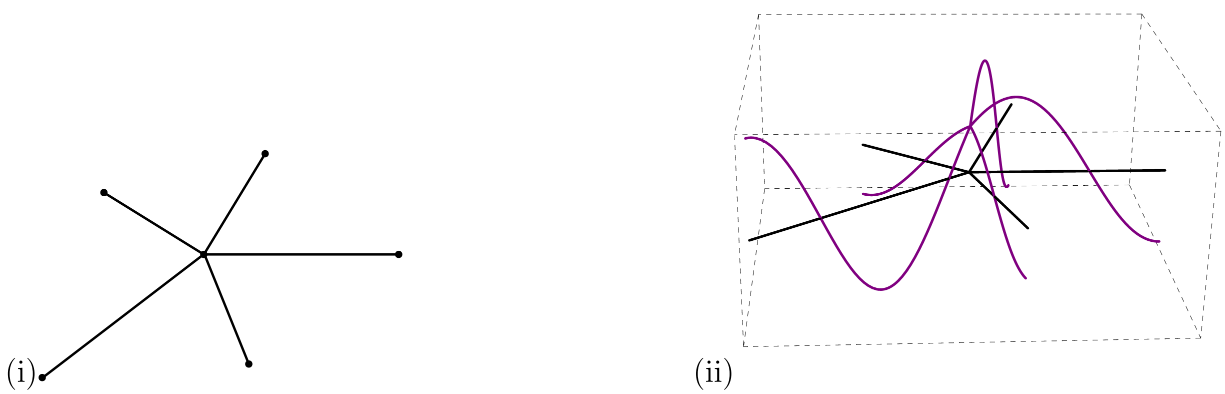

FIGURE 0.2. (i) A star graph Γ . (ii) A plot of a solution (standing wave). The graph in black, and the values of the function in purple

Denote these fractional parts by $\left[2\frac{L_j}{\lambda_n}\right] := 2\frac{L_j}{\lambda_n} - \left\lfloor 2\frac{L_j}{\lambda_n}\right\rfloor$, for each edge. We introduce the parameter

$$\rho_n := \frac{\sum_{j=1}^{E}\left[2\frac{L_j}{\lambda_n}\right]}{E} = \frac{\sum_{j=1}^{E}\left[\frac{L_j k}{\pi}\right]}{E},$$

which is the *average fractional part.*

GOALS

This project will deal with the question of "how many wave lengths can fit in a quantum graph", by investigating the *average fractional part* ρ_n. For a star graph we will show that the properties of ρ_n can be related to the function

$$\rho(\vec{x}) = \frac{\sum_{j=1}^{E} x_j}{E}$$

$$for\ \vec{x}\ that\ solve\ \sum_{j=1}^{E} \tan(x_j) = 0,$$

which we can investigate.

The project goals are

(1) Find bounds for ρ and relate them to *spectral position**.
(2) Calculate the *probability density** of ρ.
(3) Use the bounds and probability density of ρ to find *degenerate** eigenfunctions.

$*$ Will be explained in details during the project.

Requirements

Personal requirements

(1) Multi-variable calculus ("Infi 3").
(2) ODE ("Madar")
(3) Probability

Team requirements (at least one person in the team needs to know):

(1) Differential geometry.
(2) Basic knowledge in Matlab and\or Mathematica.