Numerical explorations of open problems from operator theory

The goal of this project is to explore one or several open problems that come from diverse areas in operator theory. All of these problems are related to recent research and are of real interest. In this project, the students will learn the theoretical background of **one** of the following conjectures, and will write a program that will numerically test their validity. Such experimentation might help find counter examples to the conjectures or help identify corrections to the conjectures. More interestingly, thinking about the design of numerical experiments might lead to new insights and to a better understanding of these problems.
The problems (stated as conjectures):

1. **Dimension dependent von Neumann inequality:** Fix $d, n \in \mathbb{N}$. There exists a finite constant $C_{d,n}$ such that for every polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ (in d complex variables) and every d-tuple of $T = (T_1, \dots, T_d)$ of commuting $n \times n$ matrices such that $\|T\| := \|\sum T_i T_i^*\| \leq 1$, it holds that
$$\|p(T)\| \leq C_{d,n} \sup_{|z| \leq 1} |p(z_1, \dots, z_d)|.$$
What we know: $C_{1,n} = 1$ for all n; this is known as *von Neumann's inequality*. It is also known that for $d > 1$, if the constants exist and are finite, then $C_{d,n} \to \infty$ as $n \to \infty$. We don't even know whether $C_{2,2}$ is finite and if so what it is.
2. **The complex matrix cube conjecture**: Given matrices $A_1, \dots, A_d$, such that $\|A_i\| \leq 1$ for all $i = 1, \dots, d$, there exist *commuting* normal matrices $N_1, \dots, N_d$ such that $\|N_i\| \leq \sqrt{d}$ and
$$N_i = \begin{pmatrix} A_i & * \\ * & * \end{pmatrix}$$
for all $i = 1, \dots, d$.
What we know: The conjecture is true under the assumption that $A_1, \dots, A_d$ are all selfadjoint. It is also known that one can always find such $N_1, \dots, N_d$ with $\|N_i\| \leq \sqrt{2d}$, but this constant is not known to be optimal.
3. **The stable division problem:** Let $\mathbb{C}[z_1, \dots, z_d]$ be the algebra of polynomials in d compex variables, and endow it with some reasonable norm, for example $\|\sum_\alpha c_\alpha z^\alpha\|^2 = \sum_\alpha |c_\alpha|^2$. Given an ideal I in $\mathbb{C}[z_1, \dots, z_d]$, a set $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_d]$ is said to be a **basis** for I if for every $p \in I$, there exist $g_1, \dots, g_k \in \mathbb{C}[z_1, \dots, z_d]$ such that $p = \sum g_i f_i$. The conjecture is that there exist bases "with norm control". To be precise: it says that for every homogeneous ideal I in $\mathbb{C}[z_1, \dots, z_d]$, one can always find a basis $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_d]$ and a constant C such that for every $p \in I$, there exist $g_1, \dots, g_k \in \mathbb{C}[z_1, \dots, z_d]$ such that $p = \sum g_i f_i$, and, in addition,
$$\sum \|g_i f_i\| \leq C\|p\|.$$
What we know: For $d = 2$ the conjecture holds. We also know that for a certain norm (unfortunately, not a natural one), given an ideal, one can make a change of variables after which it has a basis with the desired property.

Prerequisites: To participate in this project the student should have taken the first-year courses in linear algebra, and should already know how to program in some language with which numerical experimentation of the above problems is possible (e.g., Matlab, Mathematica, Maple, Python, etc.) Ideally, the student should also have taken a course in Functional Analysis, not so much for performing the experiments, but for understanding the theoretical background.

בדיקות נומריות של השערות מתורת האופרטורים

בפרויקט זה, הסטודנטים ילמדו את הרקע התיאורטי של השערה מסוימת מתורת האופרטורים, ויכתבו תוכנה שתבדוק את נכונות ההשערות באופן נומרי. ניסוי שכזה עשוי לגלות דוגמא נגדית או לתת כיוון להשערה יותר מדויקת. כמו כן, ישנה תקווה שתכנון של ניסוי שכזה ישפר את ההבנה שלנו את הבעיות הללו.

רקע נדרש: הרקע הנדרש ההכרחי הוא אלגברה לינארית של שנה א', וכמובן גם ניסיון בתכנות בשפה כלשהי (כגון Matlab, Mathematica, Maple, Python, וכדו'). כמו כן, רצוי להיות רקע כלשהו באנליזה פונקציונלית, בעיקר כדי להבין את הרקע התיאורטי.