Mathematics of fluids in motion

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What is the "right" way of solving PDE?

However beautiful the strategy, you should occasionally look at the results...
Sir Winston Churchill
[1874-1965]

Continuum model of a compressible viscous fluid

Compressible Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mu \Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}.$$

$$\mu > 0, \ \lambda \geq 0$$

Physical space - boundary conditions

$$\Omega \subset R^N, \ N = 1, 2, 3, \ \mathbf{u}|_{\partial\Omega} = 0$$

Initial state

$$\varrho(0, \cdot) = \varrho_0 > 0 \ \mathbf{u}(0, \cdot) = \mathbf{u}_0$$

Isentropic EOS

$$p(\varrho) = a\varrho^\gamma, \ \gamma \geq 1, \ a > 0$$

Weak solutions

Equation of continuity

$$\left[\int_\Omega \varrho\varphi \, \mathrm{d}x\right]_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\varphi \, \mathrm{d}x\mathrm{d}t$$

φ continuously differentiable

Momentum equation

$$\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi} \, \mathrm{d}x\right]_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2}\int_\Omega \Big(\varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\Big) \, \mathrm{d}x\mathrm{d}t$$

$$- \int_{t_1}^{t_2}\int_\Omega \Big(\mu\nabla_x\mathbf{u} : \nabla_x\boldsymbol{\varphi} + \lambda\mathrm{div}_x\mathbf{u}\mathrm{div}_x\boldsymbol{\varphi}\Big) \, \mathrm{d}x\mathrm{d}t.$$

$\boldsymbol{\varphi}$ continuously differentiable, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$

Discretization - numerics

Discrete time steps

$$t_0 = 0, \;\; t_{n+1} = t_n + \Delta t$$

Implicit time discretization

$$\int_\Omega \Big(\varrho_n - \varrho_{n-1} \Big) \varphi \ \mathrm{d}x \approx \Delta t \int_\Omega \varrho_n \mathbf{u}_n \cdot \nabla_x \varphi \ \mathrm{d}x$$

$$\int_\Omega \Big(\varrho_n \mathbf{u}_n - \varrho_{n-1} \mathbf{u}_{n-1} \Big) \cdot \boldsymbol{\varphi} \ \mathrm{d}x$$

$$\approx \Delta t \int_\Omega \Big(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \Big) \ \mathrm{d}x\mathrm{d}t$$

$$- \Delta t \int_\Omega \Big(\mu \nabla_x \mathbf{u}_n : \nabla_x \boldsymbol{\varphi} + \lambda \mathrm{div}_x \mathbf{u}_n \mathrm{div}_x \boldsymbol{\varphi} \Big) \ \mathrm{d}x$$

Spatial discretization - finite volume scheme

Elementary volumes

$$\Omega \approx \Omega_h = \cup_{K \in K_h} K.$$

Discrete operators

σ – the common face of two neighboring elements, $\mathbf{n}$ – the normal vector

$$[[\phi]]_\sigma = \lim_{s \to 0+} \phi(x + s\mathbf{n}) - \lim_{s \to 0+} \phi(x - s\mathbf{n})$$

$$\nabla_x \varphi \approx [[\phi]]_\sigma \mathbf{n}$$

$$\overline{r} = \frac{\lim_{s \to 0+} r(x + s\mathbf{n}) + \lim_{s \to 0+} r(x - s\mathbf{n})}{2}$$

Numerical solutions

Equation of continuity

$$\int_{\Omega_h} \frac{\varrho_n^h - \varrho_{n-1}^h}{\Delta t} \varphi \, \mathrm{d}x = \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \left(\overline{\varrho_n^h \mathbf{u}_n^h} \cdot \mathbf{n} [[\varphi]]_\sigma - \lambda_h [[\varrho^h]]_\sigma [[\varphi]]_\sigma \right) \mathrm{d}S_h$$

Momentum equation

$$\int_{\Omega_h} \frac{\varrho_n^h \mathbf{u}_n - \varrho_{n-1}^h \mathbf{u}_{n-1}^h}{\Delta t} \cdot \boldsymbol{\varphi} \, \mathrm{d}x$$

$$= \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \left(\overline{\varrho_n^h \mathbf{u}_n^h \otimes \mathbf{u}_n^h} \cdot \mathbf{n} \cdot [[\boldsymbol{\varphi}]]_\sigma + \overline{p(\varrho_n^h)} \mathbf{n} \cdot [[\boldsymbol{\varphi}]]_\sigma \right) \mathrm{d}S_h$$

$$- \frac{1}{h} \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \left(\mu [[\mathbf{u}_n^h]]_\sigma : [[\boldsymbol{\varphi}]]_\sigma + \frac{1}{h} \lambda [[\mathbf{u}_n^h]]_\sigma \cdot \mathbf{n} [[\boldsymbol{\varphi}]]_\sigma \cdot \mathbf{n} \right) \mathrm{d}S_h$$

$$- \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \lambda_h [[\varrho_n^h \mathbf{u}_n^h]]_\sigma \cdot [[\boldsymbol{\varphi}]]_\sigma \mathrm{d}S_h.$$

φ piece–wise constant, $\boldsymbol{\varphi}$ piece–wise constant, $\boldsymbol{\varphi}|_K = 0$, K boundary vol.

Numerical viscosity

Extra viscosity added

$$\sum_{\sigma\in\Sigma_h}\int_{\sigma_h}\lambda_h[[\varrho^h]]_\sigma[[\varphi]]_\sigma\Big)\mathrm{d}S_h \approx -h\mathrm{div}_x(\lambda_h\nabla_x\varrho)$$

$$\sum_{\sigma\in\Sigma_h}\int_{\sigma_h}\lambda_h[[\varrho_n^h\mathbf{u}_n^h]]_\sigma\cdot[[\boldsymbol{\varphi}]]_\sigma\mathrm{d}S_h \approx -h\mathrm{div}_x(\lambda_h\nabla_x(\varrho\mathbf{u}))$$

Synergy analysis–numerics

Claim:
Suppose the following:

- The initial data ϱ_0, $(\varrho\mathbf{u})_0$ are smooth $\varrho_0 > 0$, and $(\varrho\mathbf{u})_0$ satisfies relevant compatibility conditions
- The domains Ω_h approximate a domain Ω, where the latter has smooth boundary
- The sequence of family of approximate densities $\{\varrho_n^h\}_{h>0, n\geq 1}$, is bounded uniformly for $\Delta t, h \to 0$

Conclusion:

$$\varrho_n^h \to \varrho, \ \mathbf{u}_n^h \to \mathbf{u} \text{ in } L^1((0,T)\times\Omega),$$

where ϱ, $\mathbf{u}$ is a smooth solution of the problem

Convergence proof

Convergence to generalized - measure–valued solutions

Natural energy estimates - stability
Error of the scheme - consistency

Weak–strong uniqueness

Measure–valued and strong solutions emenating from the same initial data coincide as long as the latter exists

Local existence, blow up criterion

Strong solutions exist on a possible short time interval as long as the data and the domain are sufficiently smooth. The life span can be extended as long as the density (pressure) remains bounded

Inviscid fluids

Complete Euler system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right] + \mathrm{div}_x (p \mathbf{u}) = 0$$

Impermeability boundary conditions

$$\Omega \subset R^N, \ N = 1, 2, 3, \ \mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Gibbs's relation

$$\vartheta Ds = De + pD \left(\frac{1}{\varrho} \right)$$

Entropy balance

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) \geq 0$$

Weak solutions

Field equations

$$\left[\int_\Omega \varrho \varphi(t,\cdot)\ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right]\ \mathrm{d}x\mathrm{d}t$$

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right]\ \mathrm{d}x\mathrm{d}t$$

$$\left[\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \varphi(t,\cdot)\ \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \partial_t \varphi + \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p \right) \mathbf{u} \nabla_x \varphi \right]\ \mathrm{d}x\mathrm{d}t$$

Entropy inequality

$$\left[\int_\Omega \varrho s \varphi(t,\cdot)\ \mathrm{d}x \right]_{t=0}^{t\to\tau-} \geq \int_0^T \int_\Omega \left[\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi \right]\ \mathrm{d}x$$

Ill posedness

Theorem [EF, Ch.Klingenberg, O.Kreml, S.Markfelder, 2017]

Let $\Omega \subset R^N$ be a bounded Lipschitz domain. Let the initial data $\varrho_0 > 0$, $\vartheta_0 > 0$ be piece-wise constant functions.
Then there exists a vector field $\mathbf{u}_0 \in L^\infty(\Omega; R^N)$ such that the complete Euler system admits infinitely many solutions starting from ϱ_0, ϑ_0, $\mathbf{u}_0$. In addition, the entropy balance holds as an equality, meaning for all test functions $\varphi \in C^1([0, T] \times \overline{\Omega})$ (not necessarily non–negative).

Convex integration

Incompressible Euler system with constant pressure

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0,\ \mathrm{div}_x \mathbf{v} = 0,$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right) = 0$$

"do nothing boundary conditions"

Weak formulation

$$\int_0^\tau \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x\mathrm{d}t = 0$$

$$\left[\int_\Omega \mathbf{v} \cdot \boldsymbol{\varphi} \ \mathrm{d}\mathbf{x} \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{v} \otimes \mathbf{v} : \nabla_x \boldsymbol{\varphi} - \frac{1}{N} |\mathbf{v}|^2 \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$\boxed{\textit{for all}}$ φ, $\boldsymbol{\varphi} \in C^1([0,T] \times R^N)$

Incompressible fluid flows

DeLellis–Székelyhidi [2010]

Let $N = 2, 3$ and let

$$E = \frac{1}{2}|\mathbf{v}|^2$$

be the kinetic energy associated to the field $\mathbf{v}$. There exists $\Lambda_0 \geq 0$ such that for any $\Lambda \geq \Lambda_0$, there is $\mathbf{v}_0 \in L^\infty(\Omega; R^N)$ such that the "Euler problem" admits infinitely many solutions $\mathbf{v}$ in the class

$$\mathbf{v} \in C_{\rm weak}([0, T]; L^2(\Omega; R^N)) \cap L^\infty((0, T) \times \Omega; R^N)$$

such that

$$E = \frac{1}{2}|\mathbf{v}|^2 = \frac{1}{2}|\mathbf{v}_0|^2 = \Lambda \text{ for any } t \in [0, T].$$

Application to the full Euler system, I

Domain decomposition

$$\overline{\Omega} = \cup_{i=1} \overline{K}_i$$

$$\varrho = \varrho_i > 0, \ \vartheta = \vartheta_i > 0 \text{ on each } K_i$$

New system

$$\left[\int_{K_i} \mathbf{v}_i \cdot \boldsymbol{\varphi} \ \mathrm{d}\mathbf{x} \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_{K_i} \left[\mathbf{v}_i \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{v}_i \otimes \mathbf{v}_i}{\varrho_i} : \nabla_x \boldsymbol{\varphi} - \frac{1}{N} \frac{|\mathbf{v}_i|^2}{\varrho_i} \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$\int_0^\tau \int_{K_i} \mathbf{v}_i \cdot \nabla_x \varphi \ \mathrm{d}x\mathrm{d}t = 0$$

$$\frac{1}{2} \frac{|\mathbf{v}_i|^2}{\varrho_i} = \Lambda - \frac{N}{2} p(\varrho_i, \vartheta_i)$$

Λ independent of i

Application to the full Euler system, II

Equation of continuity

$$\mathbf{u}_i = \frac{1}{\varrho_i}\mathbf{v}_i,$$

$$\left[\int_{K_i} \varrho_i \varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{K_i} \left[\varrho_i \partial_t \varphi + \varrho_i \mathbf{u}_i \cdot \nabla_x \varphi\right] \mathrm{d}x\mathrm{d}t$$

Momentum equation

$$\left[\int_{K_i} \varrho_i \mathbf{u}_i \cdot \boldsymbol{\varphi} \ \mathrm{d}\mathbf{x}\right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_{K_i} \left[\varrho_i \mathbf{u}_i \cdot \partial_t \boldsymbol{\varphi} + \varrho_i \mathbf{u}_i \otimes \mathbf{u}_i : \nabla_x \boldsymbol{\varphi} + p(\varrho_i, \vartheta_i)\mathrm{div}_x \boldsymbol{\varphi} - \boxed{2\Lambda \mathrm{div}_x \boldsymbol{\varphi}}\right] \ \mathrm{d}x\mathrm{d}t$$

Application to the full Euler system, III

Energy balance

$$\left[\int_{K_i}\left(\frac{1}{2}\varrho_i|\mathbf{u}_i|^2+\varrho_i e(\varrho_i,\vartheta_i)\right)\varphi\,\mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$=\int_0^\tau\int_{K_i}\left[\left(\frac{1}{2}\varrho_i|\mathbf{u}_i|^2+\varrho_i e(\varrho_i,\vartheta_i)\right)\partial_t\varphi\right.$$
$$\left.+\left(\frac{1}{2}\varrho_i|\mathbf{u}_i|^2+\varrho_i e(\varrho_i,\vartheta_i)+p(\varrho_i,\vartheta_i)\right)\mathbf{u}_i\cdot\nabla_x\varphi\right]\mathrm{d}x\mathrm{d}t$$

Entropy balance

$$\left[\int_{K_i}\varrho_i s(\varrho_i,\vartheta_i)\varphi\,\mathrm{d}x\right]_{t=0}^{t=\tau}=\int_0^\tau\int_{K_i}\left[\varrho_i s(\varrho_i,\vartheta_i)\partial_t\varphi+\varrho_i s(\varrho_i,\vartheta_i)\mathbf{u}_i\cdot\nabla_x\varphi\right]\mathrm{d}x\mathrm{d}t$$

Oscillatory lemma - reformulation

Overdetermined Euler system

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right) = 0$$

Linear system vs. non-linear constitutive equation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

$$\mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I},\ \mathbb{U} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Implicit constitutive relation

$$\lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right]$$

$$\frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] \geq \frac{1}{2} |\mathbf{v}|^2$$

$$\boxed{\frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] = \frac{1}{2} |\mathbf{v}|^2} \Leftrightarrow \mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I}$$

Oscillatory lemma - subsolutions

Equations

$\mathbf{v}$, $\mathbb{U}$ smooth in $(0, T)$

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

Extremal values

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0,\ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Energy

piece-wise smooth function e

Convex set

$$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < e$$

Oscillations - De Lellis and Székelyhidi

Oscillatory increments

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0, \ \partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0$$

$$\mathbf{w}_\varepsilon, \ \mathbb{V}_\varepsilon \in C_c^\infty(Q)$$

$$\mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(Q)$$

$$\frac{N}{2} \lambda_{\max} \left[(\mathbf{v} + \mathbf{w}_\varepsilon) \otimes (\mathbf{v} + \mathbf{w}_\varepsilon) - (\mathbb{U} + \mathbb{V}_\varepsilon) \right] < e$$

Energy

$$\liminf_{\varepsilon \to 0} \int_Q \left(|\mathbf{v} + \mathbf{w}_\varepsilon|^2 \right) \geq \int_V |\mathbf{v}|^2 + c(e) \int_Q \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^\alpha$$

Isentropic (compressible) Euler system

Standard variables

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a\varrho^\gamma, \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Conservative variables

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ \mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Energy - entropy

$$E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho), \ P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma$$

$$\frac{\mathrm{d}}{\partial_t} \int_\Omega E \ \mathrm{d}x = 0 \text{ or } \int_\Omega E(\tau) \ \mathrm{d}x \boxed{\leq} \int_\Omega E(0) \ \mathrm{d}x$$

Weak solutions

Field equations

$$\left[\int_\Omega \varrho \varphi(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

$$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi}(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t, \ \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Energy inequality

$$\int_\Omega E(\tau) \ \mathrm{d}x \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x$$

$$\leq \int_\Omega E(0) \ \mathrm{d}x \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (0, \cdot) \ \mathrm{d}x \text{ for a.a. } \tau \geq 0$$

Dissipative measure–valued (DMV) solutions

Physical space $\to$ Phase space

$$\mathcal{F} = \left\{ [\varrho, \mathbf{m}] \in R^{N+1} \ \Big| \ \varrho \geq 0 \right\}$$

$$(t, x) \in Q_T \equiv [0, T] \times \Omega \mapsto \mathcal{V}_{t,x} \in L^\infty_{\rm weak}(Q_T; \mathcal{P}(\mathcal{F}));$$

Field equations

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \ {\rm d}x{\rm d}t$$

$$= - \int_\Omega < \mathcal{V}_{0,x}; \varrho > \varphi(0, \cdot) \ {\rm d}x + \int_0^T \int_{\overline{\Omega}} \nabla_x \varphi \cdot {\rm d}\mu^1_C$$

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, {\rm div}_x \boldsymbol{\varphi} \right] \ {\rm d}x{\rm d}t$$

$$= - \int_\Omega < \mathcal{V}_{0,x}; \mathbf{m} > \cdot \boldsymbol{\varphi}(0, \cdot) \ {\rm d}x + \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} \cdot {\rm d}\mu^2_C$$

Concentration measures

$\mu^1_C, \ \mu^2_C$ signed Radon measures "sitting" on the physical space $\overline{Q}_T$

Dissipation balance

Energy inequality

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \, \mathrm{d}x + \mathcal{D}(\tau) \leq \int_\Omega \left\langle \mathcal{V}_{0,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \, \mathrm{d}x$$

Compatibility conditions

there exists a constant $d > 0$ such that

$$\int_0^T \psi \int_{\overline{\Omega}} d|\mu_C^1| + \int_0^T \psi \int_{\overline{\Omega}} d|\mu_C^2| \leq d \int_0^T \psi \mathcal{D} \mathrm{d}t$$

for any $\psi \in C_c^1[0, T)$, $\psi \geq 0$

Relative energy

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)$$

$$= \frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho} - \mathbf{U}\right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)$$

$$P'(r)r - P(r) = p(r)$$

Relative energy decomposition

$$\int_\Omega \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}x =$$

$$\frac{1}{2}\int_\Omega \left[\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] \ \mathrm{d}x - \int_\Omega \mathbf{m}\cdot\mathbf{U} \,\mathrm{d}x + \int_\Omega \varrho\left[\frac{1}{2}|\mathbf{U}|^2 + P'(r)\right] \ \mathrm{d}x + \int_\Omega p(r) \,\mathrm{d}x$$

Dissipative inequality

Relative energy inequality

$$\left[\int_\Omega \mathcal{E}\left(\varrho, \mathbf{m} \;\middle|\; r, \mathbf{U}\right)\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$\leq \int_0^\tau \int_\Omega \varrho(\mathbf{U} - \mathbf{m})\partial_t \mathbf{U} + \mathbf{m}\left(\mathbf{U} - \frac{\mathbf{m}}{\varrho}\right) : \nabla_x \mathbf{U}\ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_\Omega p(\varrho)\mathrm{div}_x \mathbf{U}\ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left[\partial_t P'(r) + \mathbf{m} \cdot \nabla_x P'(r) - r\partial_t P(r)\right]\ \mathrm{d}x\mathrm{d}t$$

Test functions

$$r > 0,\ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Weak–strong uniqueness, I

Strong solution

$$\partial_t r + \mathrm{div}_x (r\mathbf{U}) = 0$$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} = -\frac{\nabla_x p(r)}{r} = -\nabla_x P'(r)$$

Relative energy inequality, step I

$$\left[\int_\Omega \mathcal{E} \left(\varrho, \mathbf{m} \ \Big| \ r, \mathbf{U} \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$\leq \int_0^\tau \int_\Omega |\nabla_x \mathbf{U}| \varrho \left| \mathbf{U} - \frac{\mathbf{m}}{\varrho} \right|^2 \ \mathrm{d}x \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \varrho (\mathbf{U} - \mathbf{m}) \cdot \nabla_x P'(r) \ \mathrm{d}x \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega p(\varrho) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left[\partial_t P'(r) + \mathbf{m} \cdot \nabla_x P'(r) - r \partial_t P(r) \right] \ \mathrm{d}x \mathrm{d}t$$

Weak–strong uniqueness, II

Renormalized equation

$$\partial_t P(r) + \mathrm{div}_x \left(P(r)\mathbf{U} \right) = -p(r)\mathrm{div}_x \mathbf{U}$$
$$P'(r)r - P(r) = p(r)$$

Relative energy inequality, step II

$$\left[\int_\Omega \mathcal{E}\left(\varrho, \mathbf{m} \,\Big|\, r, \mathbf{U} \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$\leq \int_0^\tau \int_\Omega \mathcal{E}\left(\varrho, \mathbf{m} \,\Big|\, r, \mathbf{U} \right) \ \mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau \int_\Omega \left[p(\varrho) - p'(r)(\varrho - r) - p(r) \right] \mathrm{div}_x \mathbf{U} \ \mathrm{d}x\mathrm{d}t$$

Dissipative measure–valued (DMV) solutions, revisited

Physical space → Phase space

$$\mathcal{F} = \left\{ [\varrho, \mathbf{m}] \in R^{N+1} \ \middle| \ \varrho \geq 0 \right\}$$

$$(t,x) \in Q_T \equiv [0,T] \times \Omega \mapsto \mathcal{V}_{t,x} \in L^\infty_{\rm weak}(Q_T; \mathcal{P}(\mathcal{F}));$$

Field equations

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \ {\rm d}x{\rm d}t$$

$$= - \int_\Omega < \mathcal{V}_{0,x}; \varrho > \varphi(0, \cdot) \ {\rm d}x + \int_0^T \int_{\overline{\Omega}} \nabla_x \varphi \cdot {\rm d}\mu^1_C$$

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, {\rm div}_x \boldsymbol{\varphi} \right] \ {\rm d}x{\rm d}t$$

$$= - \int_\Omega < \mathcal{V}_{0,x}; \mathbf{m} > \cdot \boldsymbol{\varphi}(0, \cdot) \ {\rm d}x + \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} \cdot {\rm d}\mu^2_C$$

Concentration measures

$\mu^1_C, \ \mu^2_C$ signed Radon measures "sitting" on the physical space $\overline{Q}_T$

Dissipation balance, revisited

Energy inequality

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \, \mathrm{d}x + \mathcal{D}(\tau) \leq \int_\Omega \left\langle \mathcal{V}_{0,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \, \mathrm{d}x$$

Compatibility conditions

there exists a constant $d > 0$ such that

$$\int_0^T \psi \int_{\overline{\Omega}} d|\mu_C^1| + \int_0^T \psi \int_{\overline{\Omega}} d|\mu_C^2| \leq d \int_0^T \psi \mathcal{D} \mathrm{d}t$$

for any $\psi \in C_c^1[0,T)$, $\psi \geq 0$