Numerical exploration of open problems in operator theory

The complex matrix cube problem

Summer projects in mathematics at the Technion

September 2, 2018

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3. Reduce the general problem to the problem for a special case.

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Can we reduce the general case to the normal commuting case?

Dilations (upper bounding the gerenal case)

Theorem

There exists a constant C, such that for all pairs of $n \times n$ matrices A, B (and all n), there are two commuting normal matrices M, N such that

$$M = \begin{pmatrix} A & * \\ * & * \end{pmatrix} \quad , \quad N = \begin{pmatrix} B & * \\ * & * \end{pmatrix}$$

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Analogously, the conjectured constant for d operators is $C = \sqrt{d}$.
We shall experiment with examples to see if this could be so.