Graphs that are determined by their spectra

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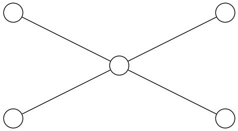
Simple Graph

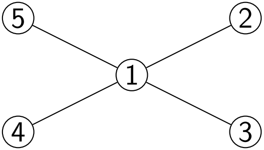
A simple graph G consists of a finite set of vertices $V(G)$ and a set of edges $E(G)$ consisting of distinct, unordered pairs of vertices.



Adjacency matrix

For a graph G on n vertices, the Adjacency matrix $A(G) = (a_{ij})$ of G is the square matrix of order n defined by

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } i, j \text{ are connected by an edge,} \\ 0 & \text{Otherwise.} \end{cases}$$



$$, \quad A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Walks in graph

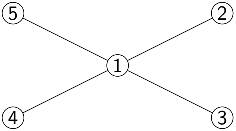
Let $V(G) = \{1, 2, \ldots, n\}$ and A be the adjacency matrix of G.
A^k_{ij}= number of walks of length k from vertex i to j.

1. $|E(G)| = \frac{1}{2}\text{trace}(A^2)$.
2. The number of triangles in G is $\frac{1}{6}\text{trace}(A^3)$.

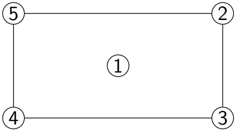
trace(A^k)= number of closed walks of length k.
Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then
$\text{trace}(A^k) = \sum_{i=1}^{n} \lambda_i^k$.

Cospectral Graphs

Cospectral Graphs: Two simple graphs G and H are cospectral if they are nonisomorphic and have same spectra.



(a)



(b)

The eigenvalues of the adjacency matrix are $\{[2]^1, [0]^3, [-2]^1\}$.

Graphs determined by the spectra

Question: Which graphs are determined by the eigenvalues of its adjacency matrix?

Example: Complete graph K_n, Cycle C_n, path P_n are determined by the spectrum of their adjacency matrix.

Laplacian matrix

$D(G) = diag(d_1, \ldots, d_n), d_i$ is the degree of i-th vertex.
Laplacian matrix: $L(G) = D(G) - A(G)$.

1. The multiplicity of zero eigenvalue of $L(G)$ is the number of connected components in G.
2. The number of spanning trees in G is equal to any cofactor of $L(G)$.

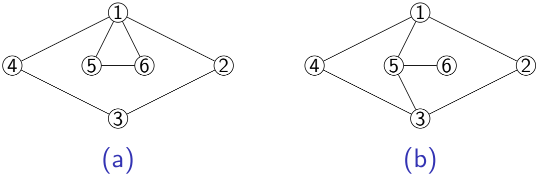


Figure: Two graphs with the same Laplacian eigenvalues.

1. Complete graph K_n.
2. Lollipop graph.

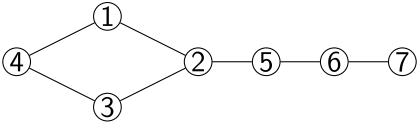


Figure: A lollipop graph.

Other matrices

1. Signless Laplacian matrix: $Q(G) = D(G) + A(G)$.
2. Normalized Laplacian matrix: $\mathcal{L}(G) = D(G)^{-\frac{1}{2}} L(G) D(G)^{-\frac{1}{2}}$.
3. Seidel matrix: $S(G) = J - I - 2A(G)$.
4. Distance matrix: (i, j)-th entry is the smallest distance between vertex i and j.

During this week we will look for known (and perhaps new) examples of co-spectral graphs and graphs that are determined by some of the matrices mentioned above.