Touchard Polynomials, Stirling Numbers and Random Permutations

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$$\begin{aligned} &|s(n,n)| = 1, \quad |s(n,1)| = (n-1)!,\ n\geq 1;\\ &|s(n+1,k)| = n|s(n,k)| + |s(n,k-1)|,\ 2\leq k\leq n. \end{aligned} \quad (2)$$

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$$(x)_n = \sum_{k=1}^{n} s(n,k)x^k. \tag{6}$$

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Thus, $x^n = \sum_{k=1}^{n} c_k$. To complete the proof we now show that $c_k = S(n,k)(x)_k$.

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For f with $|\mathrm{Im}(f)| = k$, let $\mathrm{Im}(f) = \{x_1, \cdots, x_k\}$.

Consider functions $f:[n]\to X$, where $|X|=x\in\mathbb{N}$. **Let c_k denote the number of such functions with $|\mathrm{Im}(f)|=k$.**
Proof that $c_k=S(n,k)(x)_k$:

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By (*) and (**), the matrices S and s transform between the two basis, and thus $Ss = sS = I$. □

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Example: $n = 8, m = 4$:
$0 = \sum_{j=1}^{\infty} S(8,j)s(j,4) =$
$S(8,4) - S(8,5)|s(5,4)| + S(8,6)|s(6,4)| - S(8,7)|s(7,4)| + |s(8,4)|.$

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Let X be a random variable with the distribution Poiss(λ), $\lambda > 0$:

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where B_n is the nth Bell number:
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(8) **is known as Dobiński's formula.**

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Remark. $C_1^{(n)}(\sigma)$ is the number of fixed points of σ: $C_1^{(n)} = |\{k \in [n] : \sigma_k = k\}|$. So under $P_n^{(\theta)}$, the distribution of the number of fixed points in a permutation in S_n converges as $n \to \infty$ to the Poisson distribution with parameter θ. In particular, under the uniform measure ($\theta = 1$), it converges to the Poisson distribution with parameter 1.

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That is, it is enough to show that

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(Here we use the assumption that $n \geq mj$, which insures that $n \geq kj$.)

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$$\sum_{k=1}^m \Big(\frac{1}{j}\Big)^k S(m,k)$$

$$E_n^{(1)}\big(C_j^{(n)}\big)^m = \sum_{(D_1,D_2,\cdots,D_m):|D_l|=\cdots=|D_m|=j} P_n^{(1)}\Big(\prod_{l=1}^m 1_{D_l} = 1\Big).$$

Now $\prod_{l=1}^m 1_{D_l} \not\equiv 0$ if and only if there exist a $k \in [m]$ and disjoint sets $\{A_i\}_{i=1}^k$ such that $\{D_l\}_{l=1}^m = \{A_i\}_{i=1}^k$. In this case, $\prod_{l=1}^m 1_{D_l} = \prod_{i=1}^k 1_{A_i}$, and
$P_n^{(1)}\big(\prod_{l=1}^m 1_{D_l} = 1\big) = P_n^{(1)}\big(\prod_{i=1}^k 1_{A_i} = 1\big) = \frac{\big((j-1)!\big)^k (n-kj)!}{n!}$.
The number of ways to construct k disjoint (**ordered**) sets $(A_1, \cdots, A_k)$, each with j elements from $[n]$, is
$\binom{n}{j}\binom{n-j}{j}\cdots\binom{n-(m-1)j}{j} = \frac{n!}{(j!)^k(n-jk)!}$.
Given $\{A_i\}_{i=1}^k$, the number of ways to choose $\{D_l\}_{l=1}^m$ so that $\{D_l\}_{l=1}^m = \{A_i\}_{i=1}^k$ is equal to $S(m,k)$. So

$$E_n^{(1)}\big(C_j^{(n)}\big)^m = \sum_{k=1}^m \frac{\big((j-1)!\big)^k (n-kj)!}{n!} \times \frac{n!}{(j!)^k(n-jk)!} \times S(m,k) =$$

$$\sum_{k=1}^m \big(\frac{1}{j}\big)^k S(m,k) = T_m\big(\frac{1}{j}\big)$$

$$E_n^{(1)}\big(C_j^{(n)}\big)^m = \sum_{(D_1,D_2,\cdots,D_m):|D_l|=\cdots=|D_m|=j} P_n^{(1)}\Big(\prod_{l=1}^m 1_{D_l}=1\Big).$$

Now $\prod_{l=1}^m 1_{D_l} \not\equiv 0$ if and only if there exist a $k \in [m]$ and disjoint sets $\{A_i\}_{i=1}^k$ such that $\{D_l\}_{l=1}^m = \{A_i\}_{i=1}^k$. In this case, $\prod_{l=1}^m 1_{D_l} = \prod_{i=1}^k 1_{A_i}$, and

$P_n^{(1)}\big(\prod_{l=1}^m 1_{D_l}=1\big) = P_n^{(1)}\big(\prod_{i=1}^k 1_{A_i}=1\big) = \frac{\big((j-1)!\big)^k (n-kj)!}{n!}$.

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Given $\{A_i\}_{i=1}^k$, the number of ways to choose $\{D_l\}_{l=1}^m$ so that $\{D_l\}_{l=1}^m = \{A_i\}_{i=1}^k$ is equal to $S(m,k)$. So

$$E_n^{(1)}\big(C_j^{(n)}\big)^m = \sum_{k=1}^m \frac{\big((j-1)!\big)^k (n-kj)!}{n!} \times \frac{n!}{(j!)^k(n-jk)!} \times S(m,k) =$$

$$\sum_{k=1}^m \Big(\frac{1}{j}\Big)^k S(m,k) = T_m\Big(\frac{1}{j}\Big) = \mu_{m;\frac{1}{j}}.$$