Touchard Polynomials, Stirling Numbers and Random Permutations

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Notation:

\[ [n] = \{1, 2, \cdots, n\} \]

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\[ S_6 \ni \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 5 & 3 \end{pmatrix} = (142)(36)(5), \]
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\(S_6 \ni \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 5 & 1 & 2 \end{pmatrix} = (136245) = (624513)\)
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\( \sigma_2 \) is a 6-cycle in \( S_6 \). There are 5! different 6-cycles in \( S_6 \).
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\( \sigma_2 \) is a 6-cycle in \( S_6 \). There are 5! different 6-cycles in \( S_6 \).

There are \((n - 1)!\) different \( n \)-cycles in \( S_n \).
Rising Factorials

\[ x^{(n)} := x(x+1) \cdots (x+n-1) \]
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\[ x^{(n)} := x(x+1) \cdots (x+n-1) \left( \equiv \sum_{k=1}^{n} a_{nk} x^k \right), \quad x \in \mathbb{R}, \quad n \geq 1. \quad (1) \]
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Unsigned Stirling Numbers of the First Kind:

\[ |s(n, k)| := \]

the number of permutations in \( S_n \) with exactly \( k \) cycles, \( 1 \leq k \leq n \)
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**Recurrence relation**

\[
|s(n, n)| = 1, \quad |s(n, 1)| = (n - 1)!, \quad n \geq 1;
|s(n + 1, k)| = n|s(n, k)| + |s(n, k - 1)|, \quad 2 \leq k \leq n. \tag{2}
\]
Rising Factorials

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It is easy to check that the coefficients \( \{a_{nk}\} \) in (1) also satisfy (2); thus, \( a_{nk} = |s(n, k)|. \)
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It is easy to check that the coefficients \( \{a_{nk}\} \) in (1) also satisfy (2); thus, \( a_{nk} = |s(n, k)| \). Therefore

\[ x^{(n)} = \sum_{k=1}^{n} |s(n, k)| x^k. \quad (3) \]
Falling Factorials

\[(x)_n := x(x - 1) \cdots (x - n + 1), \ x \in \mathbb{R}, \ n \geq 1. \quad (4)\]
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Substituting \(-x\) for \(x\) in (4), we have

\[(-x)_n = -x(-x - 1) \cdots (-x - n + 1) = \]
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Define \(s(n, k) := (-1)^{n-k} |s(n, k)|\). The \(s(n, k)\) are called the \textbf{Stirling Numbers of the First Kind}.
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Define \(s(n, k) := (-1)^{n-k} |s(n, k)|.\) The \(s(n, k)\) are called the **Stirling Numbers of the First Kind**. From (5) we have

\[(x)_n = \sum_{k=1}^{n} s(n, k) x^k. \tag{6}\]
Stirling Numbers of the Second Kind

\[ S(n, k) := \text{number of ways to partition the set } [n] \text{ into } k \text{ nonempty sets.} \]
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Example: \( S(4, 3) = 5: \)

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Theorem.

\[ x^n = \sum_{k=1}^{n} S(n, k)(x^k), \quad (7) \]

Proof.

Consider functions \( f : [n] \rightarrow X \), where \( |X| = x \in \mathbb{N} \).

How many such functions are there?

The answer by direct count: \( x^n \).

An alternative indirect count: For \( k = 1, 2, \ldots, n \), let \( c_k \) denote the number of such functions with \( |\text{Im}(f)| = k \). (If \( x < n \), then \( c_k = 0 \) for \( x < k \leq n \).)

So the answer by this indirect count is \( \sum_{k=1}^{n} c_k \).

Thus, \( x^n = \sum_{k=1}^{n} c_k \).

To complete the proof we now show that \( c_k = S(n, k)(x^k) \).
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$S(n, k) :=$ number of ways to partition the set $[n]$ into $k$ nonempty sets.

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\sum_{k=1}^{n} S(n, k)(x)_k = x^n, \tag{7}
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**So the answer by this indirect count is** \(\sum_{k=1}^{n} c_k\).

Thus, \(x^n = \sum_{k=1}^{n} c_k\). To complete the proof we now show that \(c_k = S(n, k)(x)_k\).
Consider functions $f : [n] \to X$, where $|X| = x \in \mathbb{N}$. Let $c_k$ denote the number of such functions with $|\text{Im}(f)| = k$. Proof that $c_k = S(n, k)(x)_k$:
Consider functions $f : [n] \rightarrow X$, where $|X| = x \in \mathbb{N}$. Let $c_k$ denote the number of such functions with $|\text{Im}(f)| = k$. 

**Proof that** $c_k = S(n, k)(x)_k$:

For $f$ with $|\text{Im}(f)| = k$, let $\text{Im}(f) = \{x_1, \cdots, x_k\}$.
Consider functions $f : [n] \rightarrow X$, where $|X| = x \in \mathbb{N}$. Let $c_k$ denote the number of such functions with $|\text{Im}(f)| = k$. 

**Proof that** $c_k = S(n, k)(x)_k$:

For $f$ with $|\text{Im}(f)| = k$, let $\text{Im}(f) = \{x_1, \ldots, x_k\}$. Then $f^{-1}(\{x_1\}), f^{-1}(\{x_2\}), \ldots, f^{-1}(\{x_k\})$ is a partition of $[n]$ into $k$ non-empty sets, which we will denote by $\{B_1, \ldots, B_k\}$. 

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Touchard Polynomials, Stirling Numbers and Random Permutations
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Now:

1. There are \( S(n, k) \) ways to choose the sets \( \{B_1, \ldots, B_k\} \);
Consider functions $f : [n] \to X$, where $|X| = x \in \mathbb{N}$. Let $c_k$ denote the number of such functions with $|\text{Im}(f)| = k$. Proof that $c_k = S(n, k)(x)_k$:

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Now:
1. There are $S(n, k)$ ways to choose the sets $\{B_1, \cdots, B_k\}$;
2. There are $(x)_k$ ways to choose the $\{x_1, \cdots, x_k\}$;
Consider functions $f : [n] \rightarrow X$, where $|X| = x \in \mathbb{N}$. **Let** $c_k$ **denote the number of such functions with** $|\text{Im}(f)| = k$.

**Proof that** $c_k = S(n, k)(x)_k$:

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Consider functions \( f : [n] \rightarrow X \), where \( |X| = x \in \mathbb{N} \). Let \( c_k \) denote the number of such functions with \( |\text{Im}(f)| = k \).

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Now:
1. There are \( S(n, k) \) ways to choose the sets \( \{B_1, \cdots, B_k\} \);
2. There are \( \binom{x}{k} \) ways to choose the \( \{x_1, \cdots, x_k\} \);
3. There are \( k! \) ways to make the correspondence between \( f^{-1}(\{x_1\}), f^{-1}(\{x_2\}), \cdots, f^{-1}(\{x_k\}) \) and \( \{B_1, \cdots, B_k\} \).

Thus \( c_k = S(n, k) \times \binom{x}{k} \times k! \).
Consider functions $f : [n] \rightarrow X$, where $|X| = x \in \mathbb{N}$. Let $c_k$ denote the number of such functions with $|\text{Im}(f)| = k$.

**Proof that** $c_k = S(n, k)(x)_k$:

For $f$ with $|\text{Im}(f)| = k$, let $\text{Im}(f) = \{x_1, \cdots, x_k\}$. Then

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**Thus** $c_k = S(n, k) \times \binom{x}{k} \times k! = S(n, k) \frac{x!}{(x-k)!}$
Consider functions $f : [n] \to X$, where $|X| = x \in \mathbb{N}$. Let $c_k$ denote the number of such functions with $|\text{Im}(f)| = k$.

**Proof that $c_k = S(n, k)(x)_k$:**

For $f$ with $|\text{Im}(f)| = k$, let $\text{Im}(f) = \{x_1, \cdots, x_k\}$. Then $f^{-1}(\{x_1\}), f^{-1}(\{x_2\}), \cdots, f^{-1}(\{x_k\})$ is a partition of $[n]$ into $k$ non-empty sets, which we will denote by $\{B_1, \cdots, B_k\}$.

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Thus $c_k = S(n, k) \times \binom{x}{k} \times k! = S(n, k) \frac{x!}{(x-k)!} = S(n, k)(x)_k$.
\[ x^n = \sum_{k=1}^{n} S(n, k)(x)_k, \quad n \geq 1 \]
\[ (x)_n = \sum_{k=1}^{n} s(n, k)x^k, \quad n \geq 1 \]
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x^n = \sum_{k=1}^{n} S(n, k)(x)_k, \quad n \geq 1
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\]

\(S(n, k)\) (Stirling number of the second kind) is the number of ways to partition \([n]\) into \(k\) nonempty subsets

\(|s(n, k)|\) (unsigned Stirling number of the first kind) is the number of permutations of \([n]\) which have \(k\) cycles

\(s(n, k) = (-1)^{n-k}|s(n, k)|\) (Stirling number of the first kind)
\[ x^n = \sum_{k=1}^{n} S(n, k)(x)_k, \quad n \geq 1 \]
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\[ s(n, k) = (-1)^{n-k}|s(n, k)| \] (Stirling number of the first kind)

Extend the definitions of \( S(n, k) \) and \( s(n, k) \) to all \( n, k \geq 1 \) by defining \( S(n, k) = s(n, k) = 0, \) for \( k > n. \)
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**Corollary.** \( Ss = sS = Id. \)
\[ x^n = \sum_{k=1}^{n} S(n, k)(x)_k, \ n \geq 1 \]
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That is, \(\sum_{j=1}^{\infty} S(n, j)s(j, m) = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{if } n \neq m. \end{cases}\)
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**Corollary.** \( Ss = sS = I_d \).

**Proof.**
\( x^n = \sum_{k=1}^{n} S(n, k)(x)_k, \ n \geq 1 \quad (*) \)

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**Proof.** Consider the real vector space \( V \) of polynomials of the form \( \sum_{k=1}^{n} c_k x^k, \ n \geq 1, c_k \in \mathbb{R}. \)
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Of course, \( B_1 := \{ x, x^2, x^3, \cdots \} \) is a basis for \( V \).
\[ x^n = \sum_{k=1}^{n} S(n, k)(x)_k, \quad n \geq 1 \quad (*) \]
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By \( (*) \), \( B_2 := \{ (x)_1, (x)_2, (x)_3, \ldots \} \) is also a basis for \( V \).
\[ x^n = \sum_{k=1}^{n} S(n,k)(x)_k, \quad n \geq 1 \quad (*) \]
\[ (x)_n = \sum_{k=1}^{n} s(n,k)x^k, \quad n \geq 1 \quad (**). \]

Extend the definitions of \( S(n,k) \) and \( s(n,k) \) to all \( n, k \geq 1 \) by defining \( S(n,k) = s(n,k) = 0 \), for \( k > n \).

Let \( S = \{S_{nk}\}_{n,k=1}^{\infty} \) denote the \( \infty \times \infty \) matrix with entries \( S_{nk} = S(n,k) \).

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By \( (*) \) and \( (**) \), the matrices \( S \) and \( s \) transform between the two basis, and thus \( Ss = sS = I. \) \qed
\( S(n, k) \) (Stirling number of the second kind) is the number of ways to partition \([n]\) into \(k\) nonempty subsets

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That is, \(\sum_{j=1}^{\infty} S(n, j)s(j, m) = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{if } n \neq m. \end{cases}\)
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Example: \( n = 8, m = 4 \):

\[ 0 = \sum_{j=1}^{\infty} S(8, j)s(j, 4) = S(8, 4) - S(8, 5)|s(5, 4)| + S(8, 6)|s(6, 4)| - S(8, 7)|s(7, 4)| + |s(8, 4)|. \]
Recalling the Poisson distribution

Let $X$ be a random variable with the distribution $\text{Pois}(\lambda), \lambda > 0$:

$$P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \; j = 0, 1, 2, \ldots.$$
Recalling the Poisson distribution

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**nth moment:** $\mu_{n;\lambda} := EX^n$
Let $X$ be a random variable with the distribution Poiss($\lambda$), $\lambda > 0$:

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**$n$th moment:** $\mu_{n;\lambda} := E X^n$

$$\mu_{n;\lambda} = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n \overset{(7)}{=} \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{n} S(n, k)(j)_k$$
Let $X$ be a random variable with the distribution $\text{Poiss}(\lambda)$, $\lambda > 0$:

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$$\mu_{n; \lambda} = \sum_{j=0}^{\infty} (e^{-\lambda} \frac{\lambda^j}{j!}) j^n (7) \sum_{j=0}^{\infty} (e^{-\lambda} \frac{\lambda^j}{j!}) \sum_{k=1}^{n} S(n, k)(j)_k =$$

$$= \sum_{j=0}^{\infty} (e^{-\lambda} \frac{\lambda^j}{j!}) \sum_{k=1}^{\min(j,n)} S(n, k)(j)_k$$
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$$\mu_{n;\lambda} = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n \quad (7) \quad \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{n} S(n, k)(j)_k =$$

$$= \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{\min(j,n)} S(n, k)(j)_k \quad \text{ (because } (j)_k = 0, \ k > j \text{)}$$
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$$= e^{-\lambda} \sum_{k=1}^{n} S(n, k) \sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k.$$
Let $X$ be a random variable with the distribution $\text{Poiss}(\lambda), \lambda > 0$:

$$P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \ j = 1, 2, \ldots.$$ 

**nth moment**: $\mu_{n;\lambda} := EX^n$

$$\mu_{n;\lambda} = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{n} S(n, k)(j)_k =$$

$$= \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \min(j, n) \sum_{k=1}^{\infty} S(n, k)(j)_k \quad \text{(because} \ (j)_k = 0, \ k > j)$$

$$= e^{-\lambda} \sum_{k=1}^{n} S(n, k) \sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k.$$ 

But $\sum_{j=1}^{\infty} \frac{\lambda^j}{j!} (j)_k = \sum_{j=k}^{\infty} \frac{\lambda^j}{(j-k)!}$
Let $X$ be a random variable with the distribution Poiss($\lambda$), $\lambda > 0$:

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$$\mu_{n;\lambda} = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{\min(j,n)} S(n, k)(j)_k = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{\min(j,n)} S(n, k)(j)_k \quad \text{(because (j)_k = 0, \ k > j)}$$

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But $\sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k = \sum_{j=k}^{\infty} (j-k)! = \lambda^k \sum_{j=k}^{\infty} \frac{\lambda^{j-k}}{(j-k)!}$
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$$\mu_{n;\lambda} = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n \overset{(7)}{=} \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{n} S(n, k)(j)_k =$$

$$= \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{\min(j, n)} S(n, k)(j)_k \quad \text{(because $(j)_k = 0, \ k > j$)}$$

$$= e^{-\lambda} \sum_{k=1}^{n} S(n, k) \sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k.$$

But $\sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k = \sum_{j=k}^{\infty} \frac{\lambda^j}{(j-k)!} = \lambda^k \sum_{j=k}^{\infty} \frac{\lambda^{j-k}}{(j-k)!} = \lambda^k \sum_{l=0}^{\infty} \frac{\lambda^l}{l!}$
Let $X$ be a random variable with the distribution $\text{Poiss}(\lambda)$, $\lambda > 0$:

$$P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \ j = 1, 2, \ldots.$$  

**$n$th moment:** $\mu_{n;\lambda} := \mathbb{E}X^n$

$$\mu_{n;\lambda} = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n \quad \overset{(7)}{=} \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{n} S(n, k)(j)_k =$$

$$= \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{\min(j,n)} S(n, k)(j)_k \quad \text{(because } (j)_k = 0, \ k > j)$$

$$= e^{-\lambda} \sum_{k=1}^{n} S(n, k) \sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k.$$  

But $\sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k = \sum_{j=k}^{\infty} \frac{\lambda^j}{(j-k)!} = \lambda^k \sum_{j=k}^{\infty} \frac{\lambda^{j-k}}{(j-k)!} = \lambda^k \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} = \lambda^k e^\lambda.$
Let $X$ be a random variable with the distribution $\text{Poiss}(\lambda)$, $\lambda > 0$:

$$P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \quad j = 1, 2, \cdots.$$  

**nth moment:** $\mu_{n;\lambda} := E X^n$

$$\mu_{n;\lambda} = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) \sum_{k=1}^{\min(j,n)} S(n, k)(j)_k =$$

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But $\sum_{j=k}^{\infty} \frac{\lambda^j}{j!} (j)_k = \sum_{j=k}^{\infty} \frac{\lambda^j}{(j-k)!} = \lambda^k \sum_{j=k}^{\infty} \frac{\lambda^{j-k}}{(j-k)!} = \lambda^k \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} = \lambda^k e^\lambda$. Thus,

$$\mu_{n;\lambda} = \sum_{k=1}^{n} S(n, k) \lambda^k.$$
Let $X$ be a random variable with the distribution Poiss($\lambda$), $\lambda > 0$:

$$P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \; j = 1, 2, \ldots$$

**n-th moment:** $\mu_{n;\lambda} := E(X^n) = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n$

$$\mu_{n;\lambda} = \sum_{k=1}^{n} S(n, k) \lambda^k$$
Let $X$ be a random variable with the distribution Poiss($\lambda$), $\lambda > 0$:

$$P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \ j = 1, 2, \ldots.$$  

**$n$th moment:** $\mu_{n;\lambda} := EX^n = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n$

$$\mu_{n;\lambda} = \sum_{k=1}^{n} S(n, k) \lambda^k$$

**Touchard Polynomial:** $T_n(x) := \sum_{k=1}^{n} S(n, k)x^k$
Let $X$ be a random variable with the distribution Poiss($\lambda$), $\lambda > 0$:

\[ P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \quad j = 1, 2, \ldots. \]

**n-th moment:** $\mu_{n;\lambda} := E X^n = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n$

\[ \mu_{n;\lambda} = \sum_{k=1}^{n} S(n, k) \lambda^k \]

**Touchard Polynomial:** $T_n(x) := \sum_{k=1}^{n} S(n, k) x^k$

\[ \mu_{n;\lambda} = T_n(\lambda) \]
Let $X$ be a random variable with the distribution $\text{Poiss}(\lambda)$, $\lambda > 0$:

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**$n$th moment:** $\mu_{n;\lambda} := EX^n = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n$

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**Touchard Polynomial:** $T_n(x) := \sum_{k=1}^{n} S(n, k)x^k$

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(Compare $T_n$ with (6).)
Let $X$ be a random variable with the distribution Poiss($\lambda$), $\lambda > 0$:

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**Touchard Polynomial:** $T_n(x) := \sum_{k=1}^{n} S(n, k)x^k$

$\mu_{n;\lambda} = T_n(\lambda)$

(Compare $T_n$ with (6).)

$$\sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n = EX^n = \mu_{n;\lambda} = T_n(\lambda) := \sum_{k=1}^{n} S(n, k) \lambda^k$$
Let $X$ be a random variable with the distribution $\text{Poiss}(\lambda)$, $\lambda > 0$:

$$P(X = j) = e^{-\lambda} \frac{\lambda^j}{j!}, \ j = 1, 2, \ldots .$$

**$n$th moment:** $\mu_{n;\lambda} := E X^n = \sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n$

$$\mu_{n;\lambda} = \sum_{k=1}^{n} S(n, k) \lambda^k$$

**Touchard Polynomial:** $T_n(x) := \sum_{k=1}^{n} S(n, k) x^k$

$$\mu_{n;\lambda} = T_n(\lambda)$$

(Compare $T_n$ with (6).)

$$\sum_{j=0}^{\infty} \left( e^{-\lambda} \frac{\lambda^j}{j!} \right) j^n = E X^n = \mu_{n;\lambda} = T_n(\lambda) := \sum_{k=1}^{n} S(n, k) \lambda^k$$

When $\lambda = 1$:

$$\frac{1}{e} \sum_{j=0}^{\infty} \frac{j^n}{j!} = E X^n = \mu_{n;1} = T_n(1) := \sum_{k=1}^{n} S(n, k)$$
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**where $B_n$ is the $n$th Bell number:**

$B_n = \text{the total number of partitions of } [n]$.
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where $B_n$ is the $n$th Bell number:

$B_n =$ the total number of partitions of $[n]$.

(8) is known as Dobiński’s formula.
Random Permutations

Uniform probability measure on $S_n$:
$$P_n(\sigma) := \frac{1}{n!}, \sigma \in S_n$$

We now define a family of non-uniform probability measures on $S_n$:

Let $k_n(\sigma)$ = the number of cycles in $\sigma$.

For $\theta > 0$, define the following probability measure on $S_n$:
$$P(\theta)_n(\sigma) = \theta^{k_n(\sigma)} N_n(\theta),$$
where $N_n(\theta) = \sum_{\sigma \in S_n} \theta^{k_n(\sigma)}$ is the normalizing constant.

We have
$$N_n(\theta) = \sum_{\sigma \in S_n} \theta^{k_n(\sigma)} = \sum_{k=1}^{n} \theta^{|s(n,k)|} (3),$$

Thus,
$$P(\theta)_n(\sigma) = \theta^{k_n(\sigma)} \theta^{|s(n,k)|}, \sigma \in S_n.$$

$\theta > 1$: the measure favors permutations with many cycles
$\theta \in (0,1)$: the measure favors permutations with few cycles
$\theta = 1$: uniform measure
Random Permutations

Uniform probability measure on $S_n$: $P_n(\sigma) := \frac{1}{n!}$, $\sigma \in S_n$
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We have

$N_n,\theta = \sum_{n k_1 = 1} \theta^{k_1} |s(n,k_1)| (3) = \theta^n |s(n,k_1)|$.

Thus,

$P(\theta)_n(\sigma) = \frac{\theta^{k_n(\sigma)}}{\theta^n |s(n,k_1)|}$, $\sigma \in S_n$.

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We have $N_{n,\theta} = \sum_{\sigma \in S_n} \theta^{k_n(\sigma)} = \sum_{k=1}^{n} \theta^k |s(n, k)|$
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We have $N_{n,\theta} = \sum_{\sigma \in S_n} \theta^{k_n(\sigma)} = \sum_{k=1}^{n} \theta^k |s(n, k)| \overset{(3)}{=} \theta(n)$. 

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Random Permutations

**Uniform probability measure on** $S_n$: $P_n(\sigma) := \frac{1}{n!}, \ \sigma \in S_n$

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\[ P_n^{(\theta)}(\sigma) = \frac{\theta^{kn(\sigma)}}{\theta(n)}, \quad \sigma \in S_n. \]

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\( \theta > 1 \): the measure favors permutations with many cycles

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Let \( C_j^{(n)}(\sigma) = \) the number of \( j \)-cycles in \( \sigma \in S_n, \ j = 1, \cdots, n. \)
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Let \( C_j^{(n)}(\sigma) \) = the number of \( j \)-cycles in \( \sigma \in S_n, j = 1, \ldots, n. \)

Then for each \( \theta > 0 \), we can think of \( \{C_j^{(n)}\}_{j=1}^n \) as random variables on the probability space \((S_n, P_n^{(\theta)})\).
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**Theorem.** For any \( j \), the distribution of \( C_j^{(n)} \) under \( P_n^{(\theta)} \) converges weakly to the distribution \( \text{Poiss}(\frac{\theta}{j}) \) as \( n \to \infty \).
\[ P_n^{(\theta)}(\sigma) = \frac{\theta^{kn(\sigma)}}{\theta(n)}, \quad \sigma \in S_n. \]

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\[
\lim_{n \to \infty} P_n^{(\theta)}(C_j^{(n)} = m) = e^{-\frac{\theta}{j}} \frac{(\frac{\theta}{j})^m}{m!}, \quad \text{for} \quad m = 0, 1, \ldots.
\]
\[
P_n^{(\theta)}(\sigma) = \frac{\theta^{k_n(\sigma)}}{\theta(n)}, \quad \sigma \in S_n.
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Let \(C_j^{(n)}(\sigma) = \) the number of \(j\)-cycles in \(\sigma \in S_n\), \(j = 1, \cdots, n\).

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\[
\lim_{n \to \infty} P_n^{(\theta)}(C_j^{(n)} = m) = e^{-\frac{\theta}{j}} \frac{(\frac{\theta}{j})^m}{m!}, \text{ for } m = 0, 1, \cdots.
\]

**Remark.** \(C_1^{(n)}(\sigma)\) is the number of fixed points of \(\sigma\):
\(C_1^{(n)} = |\{k \in [n] : \sigma_k = k\}|. \) So under \(P_n^{(\theta)}\), the distribution of the number of fixed points in a permutation in \(S_n\) converges as \(n \to \infty\) to the Poisson distribution with parameter \(\theta\). In particular, under the uniform measure \((\theta = 1)\), it converges to the Poisson distribution with parameter 1.
Theorem. For any $j$, the distribution of $C_j^{(n)}$ under $P_n^{(\theta)}$ converges weakly to the distribution $\text{Poiss}(\theta_j)$ as $n \to \infty$. That is,
\[
\lim_{n \to \infty} P_n^{(\theta)}(C_j^{(n)} = m) = e^{-\theta_j} \frac{(\theta_j)^m}{m!}, \text{ for } m = 0, 1, \cdots.
\]
Proof for $\theta = 1$ (the same method works with virtually no extra work for all $\theta$).
**Theorem.** For any $j$, the distribution of $C_j^{(n)}$ under $P_n^{(\theta)}$ converges weakly to the distribution Poiss($\frac{\theta}{j}$) as $n \to \infty$. That is,

$$\lim_{n \to \infty} P_n^{(\theta)}(C_j^{(n)} = m) = e^{-\frac{\theta}{m}} \frac{(\frac{\theta}{j})^m}{m!}, \text{ for } m = 0, 1, \ldots.$$ 

**Proof for $\theta = 1$ (the same method works with virtually no extra work for all $\theta$).**

**Method of moments:** It is enough to show that for all $m$, the $m$th moment of $C_j^{(n)}$ converges to the $m$th moment of the distribution Poiss($\frac{1}{j}$).
Theorem. For any $j$, the distribution of $C_j^{(n)}$ under $P_n^{(\theta)}$ converges weakly to the distribution $\text{Poiss}(\frac{\theta}{j})$ as $n \to \infty$. That is,
$$\lim_{n \to \infty} P_n^{(\theta)}(C_j^{(n)} = m) = e^{-\frac{\theta}{m} \left(\frac{\theta}{j}\right)^m}, \text{ for } m = 0, 1, \ldots.$$  

Proof for $\theta = 1$ (the same method works with virtually no extra work for all $\theta$).

Method of moments: It is enough to show that for all $m$, the $m$th moment of $C_j^{(n)}$ converges to the $m$th moment of the distribution $\text{Poiss}(\frac{1}{j})$.

That is, it is enough to show that
$$\lim_{n \to \infty} E_n^{(1)}(C_j^{(n)})^m = \mu_{m;\frac{1}{j}} = T_m\left(\frac{1}{j}\right).$$
Proof that \( \lim_{n \to \infty} E_n^{(1)}(C_j^{(n)})^m = \mu_{m;\frac{1}{j}} = T_m\left(\frac{1}{j}\right) \).
Proof that $\lim_{n \to \infty} E_n^{(1)}(C_j^{(n)})^m = \mu_{m; \frac{1}{j}} = T_m(\frac{1}{j})$.

We will show that

$$E_n^{(1)}(C_j^{(n)})^m = \mu_{m; \frac{1}{j}} = T_m(\frac{1}{j}), \text{ for } n \geq mj.$$
**Proof that** \( \lim_{n \to \infty} E_n^{(1)} (C_j^{(n)})^m = \mu_{m; \frac{1}{j}} = T_m(\frac{1}{j}) \).

We will show that

\[
E_n^{(1)} (C_j^{(n)})^m = \mu_{m; \frac{1}{j}} = T_m(\frac{1}{j}), \text{ for } n \geq mj.
\]

For \( D \subset [n] \) with \( |D| = j \), define

\[
1_D(\sigma) = \begin{cases} 
1, & \text{if } \sigma \text{ has a cycle consisting of the elements of } D; \\
0, & \text{otherwise}.
\end{cases}
\]
Proof that $\lim_{n \to \infty} E_n^{(1)} (C_j^{(n)})^m = \mu_m; \frac{1}{j} = T_m(\frac{1}{j})$.

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0, & \text{otherwise}. 
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Then $C_j^{(n)}(\sigma) = \sum_{D \subset [n]:|D|=j} 1_D(\sigma)$
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Then \( C_j^{(n)}(\sigma) = \sum_{D \subset [n] : |D| = j} 1_D(\sigma) \) and

\[
(C_j^{(n)}(\sigma))^m = \left( \sum_{D_1 \subset [n] : |D_1| = j} 1_{D_1}(\sigma) \right) \cdots \left( \sum_{D_m \subset [n] : |D_m| = j} 1_{D_m}(\sigma) \right) =
\]

\[
\sum_{(D_1, D_2, \ldots, D_m) : |D_1| = \cdots = |D_m| = j} \prod_{l=1}^m 1_{D_l}(\sigma).
\]
Proof that \( \lim_{n \to \infty} E_n^{(1)} (C_j^{(n)})^m = \mu_{m; \frac{1}{j}} = T_m \left( \frac{1}{j} \right) \).

We will show that

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\sum \prod_{l=1}^{m} 1_{D_l}(\sigma).
\]

So

\[ E_n^{(1)} (C_j^{(n)})^m = \sum (D_1, D_2, \ldots, D_m): |D_l| = \cdots = |D_m| = j P_n^{(1)} \left( \prod_{l=1}^{m} 1_{D_l} = 1 \right). \]
\[ E_n^{(1)} \left( C_j^{(n)} \right)^m = \sum_{(D_1, D_2, \ldots, D_m) : |D_1| = \cdots = |D_m| = j} P_n^{(1)} \left( \prod_{l=1}^{m} 1_{D_l} = 1 \right). \]
\[ E_n^{(1)}(C_j^{(n)})^m = \sum_{(D_1,D_2,\ldots,D_m):|D_1|=\cdots=|D_m|=j} P_n^{(1)}\left(\prod_{l=1}^m 1_{D_l} = 1\right). \]

Now \( \prod_{l=1}^m 1_{D_l} \neq 0 \).
\[ E_n^{(1)}(C_j^{(n)})^m = \sum_{(D_1,D_2,\ldots,D_m):|D_l|=\cdots=|D_m|=j} P_n^{(1)} \left( \prod_{l=1}^m 1_{D_l} = 1 \right). \]

Now \( \prod_{l=1}^m 1_{D_l} \neq 0 \), (that is, there exists some \( \sigma \in D_n \) such that \( \prod_{l=1}^m 1_{D_l}(\sigma) \neq 0 \)).
\[ E_n^{(1)}(C_j^{(n)})^m = \sum_{(D_1, D_2, \ldots, D_m) : |D_1| = \cdots = |D_m| = j} P_n^{(1)}(\prod_{l=1}^m 1_{D_l} = 1). \]

Now \( \prod_{l=1}^m 1_{D_l} \neq 0 \), (that is, there exists some \( \sigma \in D_n \) such that \( \prod_{l=1}^m 1_{D_l}(\sigma) \neq 0 \), if and only if for each pair \( i_1, i_2 \in [n] \), either \( D_{i_1} = D_{i_2} \) or \( D_{i_1} \cap D_{i_2} = \emptyset \).
\[ E_n^{(1)}(C_j^{(n)})^m = \sum_{(D_1,D_2,\ldots,D_m):|D_1|=\cdots=|D_m|=j} P_n^{(1)}\left(\prod_{l=1}^{m} 1_{D_l} = 1\right). \]

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That is, if and only if there exist a \( k \in [m] \) and disjoint sets \( \{A_i\}_{i=1}^{k} \) such that \( \{D_l\}_{l=1}^{m} = \{A_i\}_{i=1}^{k} \).
\[ E_n^{(1)}(C_j^{(n)})^m = \sum_{(D_1,D_2,\ldots,D_m) : |D_1|=\cdots=|D_m|=j} P_n^{(1)} \left( \prod_{l=1}^m 1_{D_l} = 1 \right). \]

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In this case, \( \prod_{l=1}^m 1_{D_l} = \prod_{i=1}^k 1_{A_i} \).
\[
E_n^{(1)}(C_j^{(n)})^m = \sum_{(D_1, D_2, \ldots, D_m) : |D_1| = \ldots = |D_m| = j} P_n^{(1)}\left(\prod_{l=1}^{m} 1_{D_l} = 1\right).
\]

Now \(\prod_{l=1}^{m} 1_{D_l} \neq 0\), (that is, there exists some \(\sigma \in D_n\) such that \(\prod_{l=1}^{m} 1_{D_l}(\sigma) \neq 0\), if and only if for each pair \(i_1, i_2 \in [n]\), either \(D_{i_1} = D_{i_2}\) or \(D_{i_1} \cap D_{i_2} = \emptyset\).

That is, if and only if there exist a \(k \in [m]\) and disjoint sets \(\{A_i\}_{i=1}^{k}\) such that \(\{D_l\}_{l=1}^{m} = \{A_i\}_{i=1}^{k}\).

In this case, \(\prod_{l=1}^{m} 1_{D_l} = \prod_{i=1}^{k} 1_{A_i}\), and

\[
P_n^{(1)}\left(\prod_{l=1}^{m} 1_{D_l} = 1\right) = P_n^{(1)}\left(\prod_{i=1}^{k} 1_{A_i} = 1\right) = \frac{(j-1)!^k (n-kj)!}{n!}.
\]
\[ E_n^{(1)}(C_j^{(n)})^m = \sum_{(D_1,D_2,\ldots,D_m) : |D_1|=\cdots=|D_m|=j} P_n^{(1)}(\prod_{l=1}^{m} 1_{D_l} = 1). \]

Now \( \prod_{l=1}^{m} 1_{D_l} \neq 0 \), (that is, there exists some \( \sigma \in D_n \) such that \( \prod_{l=1}^{m} 1_{D_l}(\sigma) \neq 0 \)), if and only if for each pair \( i_1, i_2 \in [n] \), either \( D_{i_1} = D_{i_2} \) or \( D_{i_1} \cap D_{i_2} = \emptyset \). That is, if and only if there exist a \( k \in [m] \) and disjoint sets \( \{A_i\}_{i=1}^{k} \) such that \( \{D_l\}_{l=1}^{m} = \{A_i\}_{i=1}^{k} \).

In this case, \( \prod_{l=1}^{m} 1_{D_l} = \prod_{i=1}^{k} 1_{A_i} \), and

\[ P_n^{(1)}(\prod_{l=1}^{m} 1_{D_l} = 1) = P_n^{(1)}(\prod_{i=1}^{k} 1_{A_i} = 1) = \frac{(j-1)!^k (n-kj)!}{n!}. \]

(Here we use the assumption that \( n \geq mj \), which insures that \( n \geq kj \).)
\[ E_{n}^{(1)}(C_{j}^{(n)})^{m} = \sum (D_{1}, D_{2}, \ldots, D_{m} : |D_{1}| = \ldots = |D_{m}| = j) \cdot P_{n}^{(1)}\left(\prod_{j=1}^{m} 1_{D_{j}} = 1\right). \]

Now \( \prod_{j=1}^{m} 1_{D_{j}} \neq 0 \) if and only if there exist a \( k \in [m] \) and disjoint sets \( \{A_{i}\}_{i=1}^{k} \) such that \( \{D_{i}\}_{i=1}^{m} = \{A_{i}\}_{i=1}^{k} \). In this case,
\[
\prod_{j=1}^{m} 1_{D_{j}} = \prod_{i=1}^{k} 1_{A_{i}}, \quad \text{and} \quad P_{n}^{(1)}\left(\prod_{j=1}^{m} 1_{D_{j}} = 1\right) = P_{n}^{(1)}\left(\prod_{i=1}^{k} 1_{A_{i}} = 1\right) = \frac{(j-1)! \cdot (n-kj)!}{n!}.
\]
\[ E_n^{(1)} (C_j^{(n)})^m = \sum (D_1, D_2, \ldots, D_m) : |D_l| = \cdots = D_m = j \, P_n^{(1)} \left( \prod_{l=1}^m 1_{D_l} = 1 \right). \]

Now, \( \prod_{l=1}^m 1_{D_l} \neq 0 \) if and only if there exist a \( k \in [m] \) and disjoint sets \( \{A_i\}_{i=1}^k \) such that \( \{D_l\}_{l=1}^m = \{A_i\}_{i=1}^k \). In this case, \( \prod_{l=1}^m 1_{D_l} = \prod_{i=1}^k 1_{A_i} \), and

\[ P_n^{(1)} \left( \prod_{l=1}^m 1_{D_l} = 1 \right) = P_n^{(1)} \left( \prod_{i=1}^k 1_{A_i} = 1 \right) = \frac{(j-1)!^k (n-kj)!}{n!}. \]

The number of ways to construct \( k \) disjoint (ordered) sets \( (A_1, \ldots, A_k) \), each with \( j \) elements from \([n]\), is

\[ \frac{n!}{(j!)^k (n-jk)!}. \]
\[ E_n^{(1)} \left( C_j^{(n)} \right)^m = \sum (D_1, D_2, \ldots, D_m) : |D_l| = \cdots = |D_m| = j \cdot P_n^{(1)} \left( \prod_{l=1}^{m} 1D_l = 1 \right). \]

Now \( \prod_{l=1}^{m} 1D_l \neq 0 \) if and only if there exist a \( k \in [m] \) and disjoint sets \( \{A_i\}_{i=1}^{k} \) such that \( \{D_l\}_{l=1}^{m} = \{A_i\}_{i=1}^{k} \). In this case, \( \prod_{l=1}^{m} 1D_l = \prod_{i=1}^{k} 1A_i \), and

\[ P_n^{(1)} \left( \prod_{l=1}^{m} 1D_l = 1 \right) = P_n^{(1)} \left( \prod_{i=1}^{k} 1A_i = 1 \right) = \frac{(j-1)!^k (n-kj)!}{n!}. \]

The number of ways to construct \( k \) disjoint (ordered) sets \((A_1, \ldots, A_k)\), each with \( j \) elements from \([n]\), is

\[ \binom{n}{j} \binom{n-j}{j} \cdots \binom{n-(m-1)j}{j} = \frac{n!}{(j!)^k (n-jk)!}. \]

Given \( \{A_i\}_{i=1}^{k} \), the number of ways to choose \( \{D_l\}_{l=1}^{m} \) so that \( \{D_l\}_{l=1}^{m} = \{A_i\}_{i=1}^{k} \) is equal to \( S(m, k) \).
\[ E_{n}^{(1)}(C_{j}^{(n)})^{m} = \sum(D_{1},D_{2},\ldots,D_{m})|D_{l}|=\cdots=D_{m}|=j P_{n}^{(1)}\left( \prod_{l=1}^{m} 1_{D_{l}} = 1 \right). \]

Now \( \prod_{l=1}^{m} 1_{D_{l}} \not\equiv 0 \) if and only if there exist a \( k \in [m] \) and disjoint sets \( \{A_{i}\}_{i=1}^{k} \) such that \( \{D_{l}\}_{l=1}^{m} = \{A_{i}\}_{i=1}^{k} \). In this case, \( \prod_{l=1}^{m} 1_{D_{l}} = \prod_{i=1}^{k} 1_{A_{i}} \), and
\[
P_{n}^{(1)}(\prod_{l=1}^{m} 1_{D_{l}} = 1) = P_{n}^{(1)}(\prod_{i=1}^{k} 1_{A_{i}} = 1) = \frac{(j-1)!}{n!} \frac{(n-kj)!}{(j!)^{k}(n-jk)!}.
\]
The number of ways to construct \( k \) disjoint (ordered) sets \( (A_{1},\ldots,A_{k}) \), each with \( j \) elements from \( [n] \), is
\[
\binom{n}{j} \binom{n-j}{j} \cdots \binom{n-(m-1)j}{j} = \frac{n!}{(j!)^{k}(n-jk)!}.
\]
Given \( \{A_{i}\}_{i=1}^{k} \), the number of ways to choose \( \{D_{l}\}_{l=1}^{m} \) so that \( \{D_{l}\}_{l=1}^{m} = \{A_{i}\}_{i=1}^{k} \) is equal to \( S(m,k) \). So
\[
E_{n}^{(1)}(C_{j}^{(n)})^{m} = \sum_{k=1}^{m} \frac{(j-1)!}{n!} \frac{(n-kj)!}{(j!)^{k}(n-jk)!} \times S(m,k).
\]

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\[ E_n^{(1)} \left( C_j^{(n)} \right)^m = \sum \left( D_1, D_2, \ldots, D_m \right) : |D_1| = \cdots = |D_m| = j P_n^{(1)} \left( \prod_{l=1}^m 1_{D_l} = 1 \right). \]

Now \( \prod_{l=1}^m 1_{D_l} \neq 0 \) if and only if there exist a \( k \in [m] \) and disjoint sets \( \{ A_i \}_{i=1}^k \) such that \( \{ D_l \}_{l=1}^m = \{ A_i \}_{i=1}^k \). In this case, \( \prod_{l=1}^m 1_{D_l} = \prod_{i=1}^k 1_{A_l} \), and

\[ P_n^{(1)} \left( \prod_{l=1}^m 1_{D_l} = 1 \right) = P_n^{(1)} \left( \prod_{i=1}^k 1_{A_l} = 1 \right) = \frac{(j-1)!}{n!} \frac{(n-kj)!}{(j!)^k (n-jk)!}. \]

The number of ways to construct \( k \) disjoint (ordered) sets \( (A_1, \ldots, A_k) \), each with \( j \) elements from \([n]\), is

\[ \binom{n}{j} \binom{n-j}{j} \cdots \binom{n-(m-1)j}{j} = \frac{n!}{(j!)^k (n-jk)!}. \]

Given \( \{ A_i \}_{i=1}^k \), the number of ways to choose \( \{ D_l \}_{l=1}^m \) so that \( \{ D_l \}_{l=1}^m = \{ A_i \}_{i=1}^k \) is equal to \( S(m, k) \). So

\[ E_n^{(1)} \left( C_j^{(n)} \right)^m = \sum_{k=1}^m \left( \frac{(j-1)!}{n!} \frac{(n-kj)!}{(j!)^k (n-jk)!} \right) \sum_{k=1}^m \left( \frac{1}{j} \right)^k S(m, k) = \]
\[ E_n^{(1)}(C_j^{(n)})^m = \sum (D_1,D_2,\ldots,D_m) : |D_1| = \cdots = |D_m| = j \ P_n^{(1)}\left( \prod_{l=1}^m 1_{D_l} = 1 \right). \]

Now \( \prod_{l=1}^m 1_{D_l} \not\equiv 0 \) if and only if there exist a \( k \in [m] \) and disjoint sets \( \{A_i\}_{i=1}^k \) such that \( \{D_l\}_{l=1}^m = \{A_i\}_{i=1}^k \). In this case, \( \prod_{l=1}^m 1_{D_l} = \prod_{i=1}^k 1_{A_i} \), and

\[ P_n^{(1)}\left( \prod_{l=1}^m 1_{D_l} = 1 \right) = P_n^{(1)}\left( \prod_{i=1}^k 1_{A_i} = 1 \right) = \frac{(j-1)!}{n!} \frac{(n-kj)!}{(j!)^k(n-jk)!}. \]

The number of ways to construct \( k \) disjoint (ordered) sets \( (A_1, \cdots, A_k) \), each with \( j \) elements from \([n]\), is

\[ \binom{n}{j} \binom{n-j}{j} \cdots \binom{n-(m-1)j}{j} = \frac{n!}{(j!)^k(n-jk)!}. \]

Given \( \{A_i\}_{i=1}^k \), the number of ways to choose \( \{D_l\}_{l=1}^m \) so that \( \{D_l\}_{l=1}^m = \{A_i\}_{i=1}^k \) is equal to \( S(m,k) \). So

\[ E_n^{(1)}(C_j^{(n)})^m = \sum_{k=1}^m \frac{(j-1)!}{n!} \frac{(n-kj)!}{(j!)^k(n-jk)!} \times \frac{n!}{(j!)^k(n-jk)!} \times S(m,k) = \sum_{k=1}^m \left( \frac{1}{j} \right)^k S(m,k) = T_m\left( \frac{1}{j} \right). \]
\[ E_n^{(1)} \left( C_j^{(n)} \right)^m = \sum \left( D_1, D_2, \ldots, D_m : |D_1| = \cdots = |D_m| = j \right) P_n^{(1)} \left( \prod_{l=1}^{m} 1_{D_l} = 1 \right). \]

Now \( \prod_{l=1}^{m} 1_{D_l} \neq 0 \) if and only if there exist a \( k \in [m] \) and disjoint sets \( \{ A_i \}_{i=1}^{k} \) such that \( \{ D_l \}_{l=1}^{m} = \{ A_i \}_{i=1}^{k} \). In this case, \( \prod_{l=1}^{m} 1_{D_l} = \prod_{i=1}^{k} 1_{A_i} \), and

\[ P_n^{(1)} \left( \prod_{l=1}^{m} 1_{D_l} = 1 \right) = P_n^{(1)} \left( \prod_{i=1}^{k} 1_{A_i} = 1 \right) = \frac{(j-1)!^k (n-kj)!}{n!}. \]

The number of ways to construct \( k \) disjoint (ordered) sets \( (A_1, \cdots, A_k) \), each with \( j \) elements from \( [n] \), is

\[ \binom{n}{j} \left( \binom{n-j}{j} \right) \cdots \left( \binom{n-(m-1)j}{j} \right) = \frac{n!}{(j!)(n-jk)!}. \]

Given \( \{ A_i \}_{i=1}^{k} \), the number of ways to choose \( \{ D_l \}_{l=1}^{m} \) so that \( \{ D_l \}_{l=1}^{m} = \{ A_i \}_{i=1}^{k} \) is equal to \( S(m, k) \). So

\[ E_n^{(1)} \left( C_j^{(n)} \right)^m = \sum_{k=1}^{m} \frac{(j-1)!^k (n-kj)!}{n!} \times \frac{n!}{(j!)(n-jk)!} \times S(m, k) = \]

\[ \sum_{k=1}^{m} \left( \frac{1}{j} \right)^k S(m, k) = T_m \left( \frac{1}{j} \right) = \mu_{m; \frac{1}{j}}. \]