בדיקות נומריות של השערות בתורת האופרטורים

בפרויקט זה, הסטודנטים ילמדו את הרקע התיאורטי של השערה מסוימת מתורת האופרטורים, ויכתבו תוכנה שתבדוק את נכונות ההשערות באופן נומרי. ניסוי שכזה עשוי לגלות דוגמא נגדית או לתת כיוון להשערה יותר מדויקת. כמו כן, ישנה תקווה שתכנון של ניסוי שכזה ישפר את ההבנה שלנו את הבעיות הללו.

רקע נדרש: הרקע הנדרש ההכרחי הוא אלגברה לינארית של שנה א׳, וכמובן גם ניסיון בתכנות בשפה כלשהי (כגון Matlab, Mathematica, Maple, Python, וכדו׳). כמו כן, רצוי להיות בעלי רקע כלשהו באנליזה פונקציונלית, בעיקר כדי להבין את הרקע התיאורטי.

Numerical explorations of open problems in operator theory

In this project, the students will learn the theoretical background of **one** of the following open problems and will write a program that will numerically test their validity. Such experimentation might help find counter examples to conjectures or help identify corrections to the conjectures. More interestingly, thinking about the design of numerical experiments might lead to new insights and to a better understanding of these problems.

Prerequisites: To participate in this project the student should have taken the first-year courses in linear algebra, and should already know how to program in some language with which numerical experimentation of the above problems is possible (e.g., Matlab, Mathematica, Maple, Python, etc.) Ideally, the student should also have taken a course in Functional Analysis, not so much for performing the experiments, but for understanding the theoretical background.

Suggested problems

1. **The complex matrix cube problem**: Find the smallest constant C_d such that the following holds: for all matrices $A_1, \dots, A_d$, such that $\|A_i\| \leq 1$ for all $i = 1, \dots, d$, there exist *commuting* normal matrices $N_1, \dots, N_d$ such that $\|N_i\| \leq C_d$ and
$$N_i = \begin{pmatrix} A_i & * \\ * & * \end{pmatrix}$$
for all $i = 1, \dots, d$.

   What we know: We used to conjecture that $C_d = \sqrt{d}$, but in last year's summer research projects, the students Mattya Ben-Efraim and Yuval Yifrach found a counter example which showed that $C_2 > \sqrt{2}$. In fact, we know by now that $C_2 > 1.543 \dots$ On the other hand, it is known that $C_d \leq \sqrt{2d}$, but this constant is not known to be optimal. We conjecture that it is. The optimal constant has been found for special classes of operators, for example for tuples of selfadjoint operators, or for tuples satisfying some kind of algebraic relation.

2. **The stable division problem:** Let $\mathbb{C}[z_1, \dots, z_d]$ be the algebra of polynomials in d compex variables, and endow it with some reasonable norm, for example $\|\sum_\alpha c_\alpha z^\alpha\|^2 = \sum_\alpha |c_\alpha|^2$. Given an ideal I in $\mathbb{C}[z_1, \dots, z_d]$, a set $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_d]$ is said to be a **basis** for I if for every $p \in I$, there exist $g_1, \dots, g_k \in \mathbb{C}[z_1, \dots, z_d]$ such that $p = \sum g_i f_i$. The conjecture is that there exist bases "with norm control". To be precise: it says that for every homogeneous ideal I in $\mathbb{C}[z_1, \dots, z_d]$, one can always find a basis $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_d]$ and a constant C such that for every $p \in I$, there exist $g_1, \dots, g_k \in \mathbb{C}[z_1, \dots, z_d]$ such that $p = \sum g_i f_i$, and, in addition,
$$\sum \|g_i f_i\| \leq C\|p\|.$$

   What we know: For $d = 2$ the conjecture holds. We also know that for a certain norm (unfortunately, not a natural one), given an ideal, one can make a change of variables after which it has a basis with the desired property.

   (This problem requires knowing also a bit of ring theory and is the most algebraic of the problems.)

3. **Dimension dependent von Neumann inequality:** Fix $d, n \in \mathbb{N}$. We wish to study the existence of a finite constant $C_{d,n}$ such that for every polynomial $p \in \mathbb{C}[z_1, \dots, z_d]$ (in d complex variables) and every d-tuple of $T = (T_1, \dots, T_d)$ of commuting $n \times n$ matrices such that $\|T\| := \|\sum T_i T_i^*\| \leq 1$, it holds that
$$\|p(T)\| \leq C_{d,n} \sup_{|z| \leq 1} |p(z_1, \dots, z_d)|.$$

   What we know: $C_{1,n} = 1$ for all n; this is known as *von Neumann's inequality*. It is also known that for $d > 1$, if the constants exist and are finite, then $C_{d,n} \to \infty$ as $n \to \infty$. We don't even know whether $C_{2,2}$ is finite and if so what it is.