PROJECTION METHODS

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In many mathematical problems motivated by real world applications (Image Reconstruction from Projections, Intensity Modulated Radiation Therapy, to name a few), the goal is to find a point, which will satisfy a given list of restrictions called constraints. When each of the constraints can be described as a closed and convex set, then such a problem is called a *Convex Feasibility Problem* (CFP).

The basic example of the CFP is a system of either linear equations $Ax = b$ or linear inequalities $Ax \leq b$. For simplicity let us have a closer look at the case with inequalities, which can be written as follows:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq & b_2 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq & b_m. \end{cases} \tag{1}$$

Solving system (1) is equivalent to finding a point in the intersection of the half-spaces (i-th inequality)

$$C_i := \{x \in \mathbb{R}^n \mid a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i\}, \tag{2}$$

where $i = 1 \ldots, m$.

In CFP the underlying assumption, notably motivated by applications, is that **one can easily find the point in C_i, which will minimize the distance between arbitrarily chosen x and C_i.** This point, denoted by $P_{C_i}x$, is called the *metric projection* of x onto C_i. For the half-space C_i defined in (2), the metric projection can even be computed explicitly using the formula

$$P_{C_i}x := x - \frac{\max\left\{0,\ \sum_{j=1}^{n} a_{ij}x_j - b_i\right\}}{\sum_{j=1}^{n} a_{ij}^2}\boldsymbol{a_i}, \tag{3}$$

where $\boldsymbol{a_i}$ is the i-th row of A. The computation of the metric projection onto C_i will always guarantee that the resulting point will satisfy at least one of the constraints, namely C_i. However, this operation won't guarantee that the resulting point will satisfy all the constraints $C_1, \ldots, C_m$. To overcome this obstacle, one could try to compute the metric projection onto the common part of the constraints $C = \bigcap_{i=1}^{m} C_i$, but at this point we reach another underlying assumption, namely, that **the projection onto the intersection of all the constraints is supposed to be difficult**; see Figure 1 (left). Roughly speaking, projection methods are developed for problems, which by their definition satisfy both mentioned assumptions and, in general, they show how to exploit the first property while avoiding the second one. An example of such an approach is illustrated in Figure 1 (right), where the graphical interpretation of the *method of alternating projections* is presented for system (1) with two inequalities.

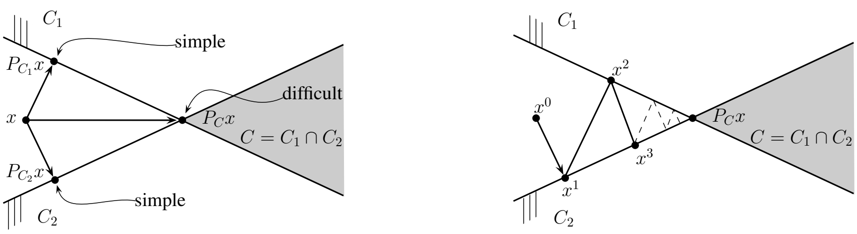

Figure 1 On the left, "simple" projections onto C_1, C_2, and "difficult" projection onto C. On the right the method of alternating projections

To summarize, in this project you will be introduced to a few basic concepts commonly used for the convergence analysis of the classical projection methods. The goal would be to apply some of these techniques to more advanced variants of projection methods and check numerically (using MATLAB or Python) whether such methods are efficient from the computational point of view. Depending on your interests, the project can be either mathematically or computationally oriented.

Introductory references for projection methods:

[1] Y. Censor, S. Zenios, *Parallel Optimization, Theory, Algorithms and Applicaions*, Oxford University Press, New York, 1997 (Section 5).

[2] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Springer, Heidelberg, 2012 (Chapter 5, Sections 5.1, 5.4-5.6).

and a MATLAB textbook:

[3] D. Hanselman, B. Littlefield, *Mastering MATLAB 7*, Pearson Education, Inc., New Jersy 2005.