The geometric mean of convex bodies

Summer Research Project in Mathematics, mentored by Liran Rotem

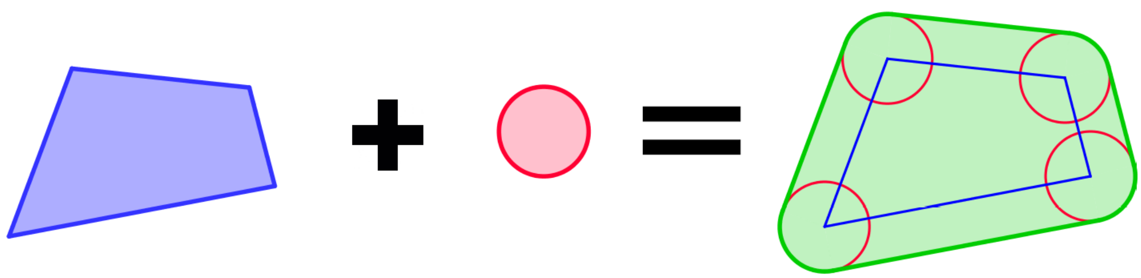
Background

A *convex body* is simply a set $K \subseteq \mathbb{R}^n$ which is convex, compact and has non-empty interior. The geometric study of such convex bodies and in particular of their volume is classical area of mathematical research, going back to the works of mathematical giants such as Minkowski, Hilbert and Alexandrov.

The *Minkowski sum* of two convex bodies $K, T \subseteq \mathbb{R}^n$ is defined in the obvious way by

$$K + T = \{x + y : \ x \in K, \ y \in T\}.$$

This is best illustrated using a picture:



Similarly, for a number $\lambda > 0$ and a convex body $K \subseteq \mathbb{R}^n$, we define $\lambda K = \{\lambda x : \ x \in K\}$, which is just a scaling of the original body K.

The famous Brunn-Minkowski inequality states that for two convex bodies $K, T \subseteq \mathbb{R}^n$ (or in fact any two Borel sets) one has

$$\left|\frac{K+T}{2}\right|^{\frac{1}{n}} \geq \frac{|K|^{\frac{1}{n}} + |T|^{\frac{1}{n}}}{2},$$

where $|\cdot|$ denotes the volume of a convex body. This inequality immediately implies the isoperimetric inequality: Let K be any convex body and let B be a ball such that $|K| = |B|$. Then $|\partial K| \geq |\partial B|$, where $|\partial K|$ denotes the surface area of K. The isoperimetric inequality, which explains why soap bubbles are round, was known already to the ancient Greeks.

In recent years several new conjectures were made regarding volumes of convex bodies. The exact details of these conjectures will not be so important to us, but they all seem to revolve about new ways to average convex bodies: Instead of considering the "arithmetic mean" $\frac{K+T}{2}$, we want to prove volume inequalities for smaller "combinations" of K and T.

A few years ago V. Milman and myself studied a new definition for the *geometric mean* of two convex bodies, which we denote by $g(K, T)$. Even though this construction does not seem to useful to prove volume inequalities, it is still a very nice geometric construction that has many nice properties. For example, like for real numbers, we have the AM-GM inequality

$$g(K, T) \subseteq \frac{K+T}{2}.$$

However, our construction is not very explicit, and not easy to compute for concrete examples. The goal of this project is to investigate some properties of the geometric mean for convex bodies.

Research Questions

There are many questions we could study in this project. Some may require computations in $\mathbb{R}^2$, some may involve computer simulations, and some may be truly deep. Here are some questions I suggest:

(i) Compute the geometric mean of some explicit convex bodies. For example, for $1 \leq p < \infty$, define

$$B_p = \left\{ x = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} |x_i|^p \leq 1 \right\}.$$

A natural problem is to compute $g(B_p, B_q)$. I have a guess for the answer, but I did not compute it even in dimension $n = 2$. There are also more complicated examples one may try and compute.

(ii) Our geometric mean $g(K,T)$ is the "unweighted" mean. In other words, you should intuitively think about $g(K,T)$ as "$\sqrt{KT}$". Is there a weighted version of the construction? In other words, can one define "$K^{1-\lambda}T^{\lambda}$" using a similar process? There is a chance that the answer will depend on one clever idea that will require nothing more than first year calculus.

As a comment, Milman and myself defined weighted geometric means in a separate paper, but using a different and more complicated process. This other definition also has several disadvantages.

(iii) Unfortunately, g does not satisfy all properties one may want. For example, it was shown by Magazinov that in general $g(\alpha K, T) \neq \sqrt{\alpha} g(K,T)$. Our paper actually contains several possible definitions for $g(K,T)$, and Magazinov only computed his 2-dimensional counterexample for one of them. I suspect that the same phenomenon happens for all versions of the construction, and this is something we can compute and check.

Of course, if you have your own questions regarding the construction, we can study those as well.

Prerequisites

As the topic is pretty elementary (which is not the same as easy!), there aren't many prerequisites. I will assume standard background in multivariable calculus and linear algebra.

We will also use a bit of functional analysis, but nothing too advanced – norms, inner products, dual norms, maybe the Hahn-Banach theorem. You may have seen most of these things in your calculus or linear algebra courses. Similarly, we will use some topological concepts, but only the very basic ones you probably saw in calculus. For example, you should know what a compact metric space is.

Obviously some background in convex geometry will be extremely useful, but this is not a part of the standard undergraduate curriculum so I am not going to assume such background. We will go over the basic definitions on the first day.

Finally, since we only have one week, I will ask you to read the relevant paper ([1]) before the first day. This is an actual research paper, so I do *not* expect you the understand everything, but you will at least have some idea of what's going on. It will also be a good way for you to tell if you find the topic interesting and if you have the necessary background.

References

[1] Vitali Milman and Liran Rotem. Non-standard constructions in convex geometry; geometric means of convex bodies. In Eric Carlen, Mokshay Madiman, and Elisabeth Werner, editors, *Convexity and Concentration*, volume 161 of *The IMA Volumes in Mathematics and its Applications*, pages 361–390. Springer, New York, NY, 2017.