Grassman Integrals and Spanning Forests

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Abstract

This project concerns the interpretation of representations for statistical mechanics models related to spanning forests: The starting point is Kirchoff's Matrix Tree Theorem. This theorem states that for a weighted graph on a set of vertices V, any determinant of the graph Laplacian can be expressed as a sum over *rooted* spanning forests and is therefore connected to the probability of sampling rooted spanning forests with probability proportional to the product of edge weights for edges in the given forest. Using Grassman variables (which is just another way of describing the algebra of differential forms) this determinant can be alternatively expressed as a fermionic Gaussian integral. It turns out that through the use of these Grassman variables, there is description available for *unrooted* spanning forests (even the there are non longer any matrices and determinants). Much less is understood about the corresponding probability measure, though one may hope that the Grassman representation can help shed led on its behavior.

A concrete benefit of the Grassman representation is that it hints that there may be further natural probabilistic structures out there: I recently showed that the unrooted spanning forest measures give a dual representation to a statistical mechanics model called the $H^{2|4}$ nonlinear sigma model. As the name suggests, the $H^{2|4}$ model is part of a whole family of models, the $H^{2|2n}$ nonlinear sigma models, $n \geq 1$. For each one, there is a dual Grassman variable representation,but a probabilistic interpretation of this representation in terms of (interacting?) spanning forests is missing (for all $n \geq 3$). In this project I will explain the previous paragraphs in some detail, and then will ask students to search for an appropriate probabilistic interpretation in the simplest nontrivial open case, when $n = 3$.

Prerequisite: First course in probability.