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UNSOLVED PROBLEMS

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C^m Interpolation Find the optimal $k^\#$ and better $C^\#$ in the finiteness theorem for $C^m(\mathbb{R}^n)$: If $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$ finite, and if for each $k^\#$-element subset $S \subset E$ the restriction $f\big|_S$ has trace norm at most 1 in $C^m(S)$, then f has trace norm at most $C^\#$ in $C^m(E)$. Pavel Shvartsman conjectures that the optimal $k^\#$ is $\prod_{i=1}^m (i+1)^{\binom{m+n-i-2}{n-2}}$. That is known to be correct for $m = 1, 2$ (all n) and for $n = 1$ (all m; here we set $\binom{-1}{-1} = 1$ and $\binom{i}{-1} = 0$ for $i \geq 0$). Current proofs of the finiteness theorem produce absurdly large $C^\#$, preventing practical applications.

$(1 + \epsilon)$ Flavor Given $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ having N points and given $\epsilon > 0$, compute $F \in C^m(\mathbb{R}^n)$ agreeing with f on E and having the least possible C^m-norm up to a factor of $(1 + \epsilon)$. Can this be done with $C(\epsilon)N \log N$ work? The answer for $C^2(\mathbb{R}^2)$ is "yes", even though the finiteness theorem fails.

Optimal Norms Compute exactly the least possible C^m norm of an interpolant for a given $f : E \to \mathbb{C}$, $E \subset \mathbb{R}^n$. Here and in the preceding question, we should allow ourselves any convenient choice of C^m norm. From Le Gruyer, Wells, Azagra & Mudarra, we understand real-valued f in $C^{1,1}(\mathbb{R}^n)$. The analogous problem for a complex-valued f is completely open.

Jet Ideals Let $\mathscr{R}$ be the ring of m-jets at 0 of functions in $C^m(\mathbb{R}^n)$. If $E \subset \mathbb{R}^n$ contains 0 as a boundary point, then let $I(E)$ be the ideal in $\mathscr{R}$ consisting of m-jets of functions that vanish on E. Which ideals in $\mathscr{R}$ arise as $I(E)$ for some E? Does every $I(E)$ already arise as $I(V)$ for a semialgebraic set V? (I think this problem is due to Nahum Zobin.)

Semialgebraic and subanalytic Whitney problems Let $f : E \to \mathbb{R}$ be a semialgebraic function defined on a semialgebraic $E \subset \mathbb{R}^n$. Suppose f extends to a C^m function on $\mathbb{R}^n$. Can we take that C^m function to be semialgebraic? Similarly, if f and E are subanalytic, can we take the C^m extension to be subanalytic?

One can formulate analogous questions in the more general setting of "bundles", i.e. families of cosets of submodules of the module of vector valued jets at x, as x varies over a semialgebraic subset of $\mathbb{R}^n$. That is related to the "Brenner-Epstein-Hochster-Kollár" problem of real algebraic geometry.

C^∞ Extension How can we tell whether a function $f : E \to \mathbb{R}$ ($E \subset \mathbb{R}^n$ compact) extends to a C^∞ function on $\mathbb{R}^n$? Examples by Wieslaw Pawłucki show that f may extend to a C^m function for every m, but not to a C^∞ function.

C^m Selection Let $E \subset \mathbb{R}^n$ be finite, and let $K(x) \subset \mathbb{R}^D$ be a given convex polytope for each $x \in E$. Compute a map $F : \mathbb{R}^n \to \mathbb{R}^D$ satisfying $F(x) \in K(x)$ for each $x \in E$ with the C^m norm of F as small as possible up to a constant factor depending only on m, n, D. If E contains N points, can the computation be done in $O(N \log N)$ operations? This is not at all understood, even though the relevant finiteness theorem is known. Even the case $m = 2$, $n = D = 1$ is open and looks hard.

Sobolev Extension The known results on extension problems in the setting of Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ hold in the range $p > n$, even though the problem makes sense in the larger range $p > \max\left\{\frac{n}{m}, 1\right\}$. What can we say when $\frac{n}{m} < p \leq n$?

Even in the range $p > n$, the known results for $W^{m,p}(\mathbb{R}^n)$ are less complete than for $C^m(\mathbb{R}^n)$. For instance, if $E \subset \mathbb{R}^n$, $\#(E) = N$, then one can compute $O(N)$ subsets S_1, S_2, ..., $S_L \subset E$, with $\#(S_l) = O(1)$ for each l such that for any $f : E \to \mathbb{R}$ we have

$$\|f\|_{C^m(E)} \sim \max_{l=1,\ldots,L} \|(f\big|_{S_l})\|_{C^m(S_l)}.$$

In the Sobolev setting, we have instead a list of $O(N)$ linear functionals $\xi_1, \ldots, \xi_L : W^{m,p}(E) \to \mathbb{R}$ such that for any $f : E \to \mathbb{R}^n$ we have

$$\|f\|_{W^{m,p}(E)} \sim \left(\sum_{l=1}^{L} |\xi_l(f)|^p\right)^{\frac{1}{p}}.$$

The ξ_l have a sparse structure, so that they can all be evaluated for a given f in $O(N)$ computer operations. However, it is not known whether by analogy with the C^m case, we can arrange that each $\xi_l(f)$ is determined by the values of f on a set S_l with $O(1)$ elements. Can one even produce a list of $O(N)$ sets S_1, S_2, ..., $S_L \subset E$, with $\#(S_l) = O(1)$ (each l), and numeric weights $\lambda_1, \ldots, \lambda_L > 0$ such that for any $f : E \to \mathbb{R}$ we have

$$\|f\|^p_{W^{m,p}(E)} \sim \sum_{l=1}^{L} \lambda_l \, \|(f\big|_{S_l})\|^p_{W^{m,p}(S_l)}?$$

These conjectures are optimistic; the analogous conjectures for linear extension operators are known to be false.

Lipschitz Selection Let (X, d) be an N-point metric space. For each $x \in X$, let $K(x) \subset \mathbb{R}^D$ be a convex polytope. How can one compute a map $F : X \to \mathbb{R}^D$ such that $F(x) \in K(x)$ for all $x \in X$, with Lipschitz norm as small as possible up to a factor $C(D)$? This is a big ill-conditioned linear programming problem. Can we do better than just applying general-purpose linear programming? How does the work of an optimal algorithm scale with the number of points N? The finiteness theorem is known to hold for such problems so perhaps one can say something. I don't know whether computational Lipschitz selection is easier, harder, or comparable to computational C^m selection.

Geometric Whitney Problems

Extrinsic Flavor Given a point cloud in a high-dimensional Euclidean space, decide whether the point cloud lies close (say, approximately within a given distance) to an embedded low-dimensional manifold with reasonable geometry (e.g. controlled curvature, volume, reach). Understand what happens when significant regions of the manifold are not close to any of the data points. Understand what happens if the data points are sampled from an unknown probability distribution on the high-dimensional space. The above questions have been studied by statisticians and computer scientists (and by Sanjoy Mitter, Hari Narayanan, and me), but the only known algorithm that is guaranteed to work requires absurdly large computer resources.

Intrinsic Flavor Decide whether a given finite metric space embeds as a fine net in a Riemannian manifold with reasonable geometry. This problem arises in medical imaging. Initial results are due to the team of Matti Lassas et al. (Matti Lassas, Hari Narayanan, Charles Fefferman, Sergei Ivanov, Slava Kurylev).