Abstract: Functional duality and Santaló type inequalities

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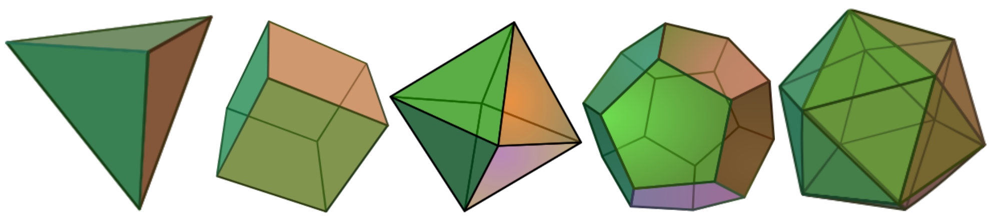
The idea of *duality* in mathematics is ancient. You may have first encountered it in linear algebra: If V is a finite dimensional vector space over $\mathbb{R}$, then its dual space is $V^* = \{f : V \to \mathbb{R} : \ f \text{ is linear}\}$. Moreover, you probably saw that to any norm $\|\cdot\|$ on V one can associate a dual norm, defined by

$$\|f\|^* = \max\left\{f(x) : \ x \in V \text{ and } \|x\| = 1\right\}.$$

We will prefer a geometric point of view on duality: To every norm $\|\cdot\|$ we can associate a convex body, its unit ball $K = \{x : \ \|x\| \le 1\}$. The polar body of K is simply the unit ball of the dual norm, $K^\circ = \left\{f : \ \|f\|^* \le 1\right\}$.

Even though the ancient greeks didn't know what a "norm" is, they had some understanding of duality. Consider for example the 5 convex bodies in $\mathbb{R}^3$ known as the *platonic solids*:



In order, these are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. The greeks already knew that the cube and the octahedron are polar to each other, so are the dodecahedron and the icosahedron, and the tetrahedron is polar to itself (Ignore for now the fact that not all platonic solids are unit balls of norms. It turns out they are "close enough").

If a body K is "big" it's polar body K° is "small", and vice versa. Therefore the *product* of their volumes, $P(K) = \mathrm{Vol}(K) \cdot \mathrm{Vol}(K^\circ)$, cannot be made too big or too small. In every dimension n there is some convex body K_{max} for which $P(K_{max})$ is maximal, and some body K_{min} for which $P(K_{min})$ is minimal. The identity of K_{max} is well-known, and the theorem identifying it is known as the Blaschke-Santaló Inequality. Can you guess what it is? The identity of K_{min} on the other hand is unknown, and is in fact a long standing open problem known as the Mahler conjecture.

This is not the problem we will attack together, as it is much too hard. Instead, we will think about *functional Santaló inequalities.* Over the last 20 years there has been a lot of activity in "Functional Geometry": The idea is to consider analytic objects, like convex functions, and think about them geometrically like some kind of "generalized convex bodies" or "generalized norms". This simple idea creates some fascinating questions in the intersection between geometry, analysis and probability.

Given a convex function $\varphi : V \to \mathbb{R}$, let us denote the polar function $\varphi^\circ : V^* \to \mathbb{R}$ by

$$\varphi^\circ(f) = \sup\left\{\frac{f(x) - 1}{\varphi(x)} : \ x \in V\right\}$$

(I am cheating a tiny bit here). This definition was first given by S. Artstein-Avidan and V. Milman around 2011. It's probably not very clear right now where this definition came from, but we will explain it. For now you can at least check to yourself that if $\varphi(x) = \|x\|$ then $\varphi^\circ(f) = \|f\|^*$.

Again, if φ is "big" in some sense then φ° is "small". We will spend most of week thinking about the functional Santaló inequality:

Problem. For which convex function φ is the product $\int e^{-\varphi} \cdot \int e^{-\varphi^\circ}$ maximal?

It may not be entirely clear why to to choose $\int e^{-\varphi}$ as a measurement of the "size" of φ. This is very common to do, but indeed it may not be the best choice. We will think about this question as well.

We will not be the first to attack this problem. S. Artstein-Avidan and B. Slomka essentially solved it when $\dim V \to \infty$: They understood which functions φ are "almost maximizers", when the dimension of the space is very big. However, finding the exact maximizer is completely open even in dimension $\dim V = 1$! We will probably start with this one dimensional case. Since this problem is probably not easy, even in dimension 1, we'll consider other related questions as well:

- Can we at least prove that a maximizer φ_{max} exists?
- If so, can we prove it's unique?
- There are some reasons to suspect that the maximizer satisfies $\varphi^\circ_{max} = \varphi_{max}$. Can we prove that?
- Can we find the maximizer is some smaller, more manageable, classes of functions?
- And of course, any other vaguely related problem you will find interesting!

This project does not require a very specialized background. In particular, no previous knowledge in convex geometry is assumed (though some will definitely be useful). You will need the usual background in multivariate calculus, some linear algebra, some basic topology (not much more than the definition of a compact topological space), and a tiny bit of functional analysis like the definition above of norms and dual norms. You will need to do some non-trivial reading before the week begins, but I'll try not to overwork you too much.