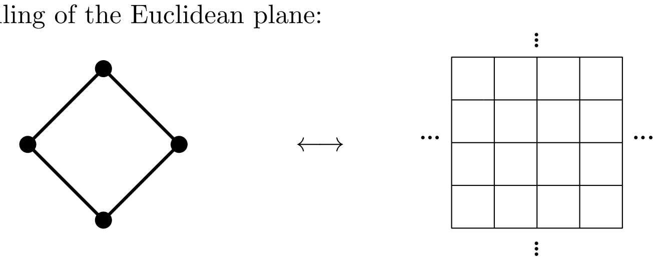
Surprising hyperbolic planes in reflection groups

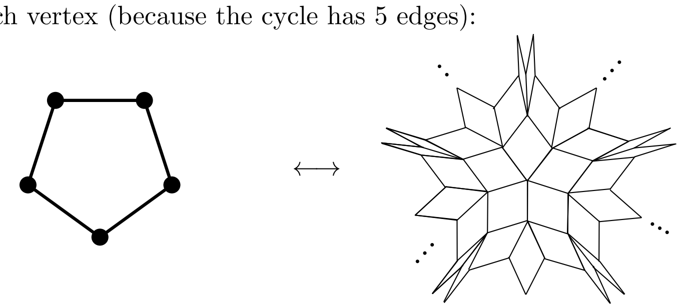
In this project you will get an idea of the mathematical field called *geometric group theory*. In this field, we study finitely generated groups by examining associated geometric objects.

Given a 4-cycle we construct an associated a group of reflections as follows: take four reflections, one for each vertex, such that two of the reflections commute if and only if there is an edge between the vertices.

The relations in the group can be visualized as squares labelled with reflections, and if we glue together squares sharing a side with the same label, we obtain a tiling of the Euclidean plane:

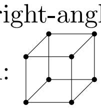


If we construct a similar square-tiling using the 5-cycle, we will not obtain a tiling of the Euclidean plane anymore. In this case, we have 5 squares glued around each vertex (because the cycle has 5 edges):



This tiling behaves like another space that is of high interest in geometric group theory called the *hyperbolic plane*. Using five reflections in the hyperbolic plane we can also associate a group of reflections to the 5-cycle.

In general, starting from a finite graph one can use this procedure to define a group, called a *right-angled Coxeter group* ("right-angled" because of the squares involved), and a square tiling, called a *Davis complex*. The geometry of the square tiling reflects the algebra of the associated group.

The goal of this project is to understand interactions between tilings of Euclidean and hyperbolic planes within a Davis complex and the algebraic implications of these interactions. In particular we will investigate the right-angled Coxeter group and Davis complex associated to the following graph:  .

Prerequisites: you should know what groups and graphs are, be comfortable with the idea of a metric space, enjoy drawing pictures and imagining complicated spaces.