One of the most interesting and compelling problems in algebraic geometry is determining whether a given $n$-dimensional algebraic variety $X$ over a field $k$ is rational (over $k$). Algebraically, this is the same as asking that the field $K(X)$ of rational functions on $X$ is purely transcendental over $k$, i.e.

$$K(X) \cong k(t_1, \ldots, t_n),$$

where $t_1, \ldots, t_n$ are algebraically independent. In general, this question is either easy or completely intractable, but for specific classes of varieties these questions are both approachable and deeply connected to other fields of mathematics, especially number theory, arithmetic geometry, and complex geometry. In this project, you will study $n$-dimensional algebraic varieties known as cubic hypersurfaces, which are defined by a single homogeneous degree $d$ polynomial $f(x_0, \ldots, x_{n+1})$ in $n+2$ variables. Specifically,

$$X = \{ [a_0, \ldots, a_{n+1}] \in \mathbb{P}^{n+1} \mid f(a_0, \ldots, a_{n+1}) = 0 \},$$

where $\mathbb{P}^{n+1}$ is the $(n+1)$-dimensional projective space defined by

$$\mathbb{P}^{n+1} := k^{n+2} / \sim, \quad [a_0, \ldots, a_{n+1}] \sim [\lambda a_0, \ldots, \lambda a_{n+1}], \quad \forall \lambda \in k^*.$$