RATIONALITY OF CUBIC HYPERSURFACES

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One of the most interesting and compelling problems in algebraic geometry is determining whether a given n-dimensional algebraic variety X over a field k is *rational* (over k). Algebraically, this is the same as asking that the field $K(X)$ of rational functions on X is purely transcendental over k, i.e.

$$K(X) \cong k(t_1, \ldots, t_n),$$

where $t_1, \ldots, t_n$ are algebraically independent. In general, this question is either easy or completely intractable, but for specific classes of varieties these questions are both approachable and deeply connected to other field of mathematics, especially number theory, arithmetic geometry, and complex geometry. In this project, you will study n-dimensional algebraic varieties known as cubic hypersurfaces, which are defined by a single homogeneous degree d polynomial $f(x_0, \ldots, x_{n+1})$ in $n+2$ variables. Specifically,

$$X = \left\{ [a_0, \ldots a_{n+1}] \in \mathbb{P}^{n+1} \;\middle|\; f(a_0, \ldots, a_{n+1}) = 0 \right\},$$

where $\mathbb{P}^{n+1}$ is the $(n+1)$-dimensional projective space defined by

$$\mathbb{P}^{n+1} := k^{n+2}/\sim, \qquad [a_0, \ldots, a_{n+1}] \sim [\lambda a_0, \ldots, \lambda a_{n+1}], \quad \forall \lambda \in k^*.$$

You will gain experience with some of the geometric and algebraic techniques involved in studying these questions by first experimenting with quadric hypersurfaces (i.e. $d = 2$), where satisfying, complete results are known. Your goal, however, will be to study cubic hypersurfaces and the question of their rationality. In increasing the degree by only one, the problem becomes disproportionately more difficult and only low dimensional cases are known. The techniques involved in the cubic case are at the forefront of current research in many subfields of algebraic geometry.

Minimal Prerequisites: Rings and Fields, Rings and Modules

Desirable Prerequisites: Modern Algebra 1