Sails and orbits

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Let X denote the space of all lattices in $\mathbb{R}^n$ of covolume 1:

$$X = \{\Lambda \subset \mathbb{R}^n : \Lambda = \mathrm{span}_{\mathbb{Z}}(v_1, \dots, v_n),\ \det(v_1 \dots v_n) = 1\}.$$

The group $G = \mathrm{SL}_n(\mathbb{R})$ acts transitively on X. In homogeneous dynamics we try to understand orbits in X of subgroups of G. A very important example is the following: Let $A < G$ be the subgroup of diagonal matrices. Given a lattice $\Lambda \in X$, how can we decide if the orbit $A\Lambda$ has a certain property we are interested in. For example, is the orbit closed? Is it compact? is it dense? Is it unbounded? All of these questions refer to the natural metric - topology on X (which we will learn about).

The projects will be focused on a certain linear-algebraic object one builds from a lattice Λ and from which one should be able to read the answers to these questions. Let $C = \{v \in \mathbb{R}^n : v_i > 0\}$ denote the positive orthant. Given a lattice Λ, we define the Klein polyhedron of Λ and the sail of Λ as follows:

$$K(\Lambda) = \mathrm{Conv}(C \cap \Lambda); \quad \mathrm{Sail}(\Lambda) = \partial K(\Lambda).$$

If you imagine things in dimension 3 (which is completely sufficient for lots of open questions) you will see that the sail of a lattice which do not contain any point on the the boundary of C is a collection of bounded convex polytopes.

Questions: Can you detect topological properties of the orbit $A\Lambda$ (such as closedness, boundedness, density etc.) from the sail of Λ? Can you write a computer program that computes and experiments with sails? Do the lattice points on a facet of a sail always span the lattice? If not, can they span a sublattice of arbitrarily large index? What polytopes appear as facets?

These are only a few examples and there are many more questions to be asked. For some the answers are known and others are open.

Prerequisites: A solid undergraduate level background in linear algebra, group theory and topology. Acquaintance with the notion of a lattice in$\mathbb{R}^n$. Good knowledge in convexity (i.e. acquaintance with the notions of polytopes, faces, facets). Advantage to students with some computational skills (i.e. ability to use a computer to test things).

Suggested reading: Read the relevant parts of the book Introduction to the geometry of numbers by Cassels. Google Klein polyhedra or Klein polyhedron and start digging.