An Optimization Framework for the Nonlinear Inverse Problems of Estimating Random Parameters in Stochastic PDEs

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Motivation. Inverse Problems

- Let $D \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary ∂D. Given maps $a : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, consider the following boundary value problem (BVP):

$$-\nabla \cdot (a(x)\nabla u(x)) = f(x), \text{ in } D, \quad u(x) = 0, \text{ on } \partial D.$$

- This simple PDE models interesting real-world phenomena and has been studied in great detail.
- For example, u may represent the *steady-state temperature* at a given point of a body; then a would be a variable thermal conductivity coefficient, and f the external heat source.
- The above BVP also models *underground steady-state aquifers* in which the parameter a is the *aquifer transmissivity coefficient*, u is the *hydraulic head*, and f is the recharge.
- In the context of the above BVP there are two problems:
  - **Direct problem**: Given a, f, compute u.
  - **Inverse problem:** Given u, f, identify a.

A Prototypical Stochastic BVP

In the applied models there exists a natural uncertainty, for example:

- Parameters are estimated based on experiments that involve noise.
- Partial (unknown) data or physical model limitations.

A sensible way is to treat these parameters as **random variables** and consider a Stochastic PDE (SPDE).

- We are given a probability space $(\Omega, \mathcal{F}, \mu)$, a bounded domain $D \subset \mathbb{R}^n$ with ∂D as its sufficiently smooth boundary.
- Given random fields $a : \Omega \times D \to \mathbb{R}$ and $f : \Omega \times D \to \mathbb{R}$, the direct problem seeks a random field $u : \Omega \times D \to \mathbb{R}$ that almost surely satisfies the following PDE with random data:

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x), \text{ in } D, \quad u(\omega, x) = 0, \text{ on } \partial D. \quad (1)$$

- Given a measurement z of u, this talk focuses on the inverse problem of estimating a so that z is closest to u in some sense.

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Outline

- A Stochastic BVP and its Variational Formulation
- Properties of the Parameter-to-Solution Map
- Optimization Formulations and Existence Results
- Finite-Dimensional Approximation
- Numerical Results

The Stochastic Variational Problem

- Given a real Banach space X, a probability space $(\Omega, \mathcal{F}, \mu)$, and an integer $p \in [1, \infty)$, the Bochner space $L^p(\Omega; X)$ consists of Bochner integrable functions $u : \Omega \to X$ with finite p-th moment, that is,

$$\|u\|_{L^p(\Omega;X)} := \left(\int_\Omega \|u(\omega)\|_X^p d\mu(\omega) \right)^{1/p} = \mathbb{E}\left[\|u(\omega)\|_X^p\right]^{1/p} < \infty.$$

- If $p = \infty$, then $L^\infty(\Omega; X)$ is the space of Bochner measurable functions $u : \Omega \to X$ such that

$$\operatorname{ess\,sup}_{\omega \in \Omega} \|u(\omega)\|_X < \infty.$$

- The variational formulation of the stochastic BVP (1) seeks $u \in V := L^2(\Omega; H_0^1(D))$ such that for all $v \in V$, we have

$$\mathbb{E}\left[\int_D a(\omega, x) \nabla u(\omega, x) \cdot \nabla v(\omega, x) dx\right] = \mathbb{E}\left[\int_D f(\omega, x) v(\omega, x)\, dx\right]. \quad (2)$$

Solvability of the Stochastic Variational Form

- We assume that there are constants k_0 and k_1 such that

$$0 < k_0 \leq a(\omega, x) \leq k_1 < \infty, \text{ almost everywhere in } \Omega \times D.$$

- Define a bilinear form $s : V \times V \mapsto \mathbb{R}$ and a functional $\ell : V \mapsto \mathbb{R}$ by

$$s(u, v) := \mathbb{E}\left[\int_D a(\omega, x)\nabla u(\omega, x) \cdot \nabla v(\omega, x) dx\right],$$
$$\ell(v) := \mathbb{E}\left[\int_D f(\omega, x) v(\omega, x)\, dx\right],$$

- The stochastic variational problem then reads: Find $u \in V$ such that

$$s(u, v) = \ell(v), \quad \text{for every } v \in V.$$

- The unique solvability follows by the Lax-Milgram lemma.
- Furthermore, there is a constant $c_1 > 0$ such that

$$\|u(\omega, x)\|_V \leq c_1 \|f(\omega, x)\|_{L^2(\Omega; H^1(D)^*)}.$$

Lipschitz continuity of the parameter-to-solution map

Theorem

For any $a(\omega, x) \in A$, the map $a(\omega, x) \mapsto u_a(\omega, x)$ is Lipschitz continuous.

Proof.

Let $u_a(\omega, x) \in V$ be the solution of (2) for $a(\omega, x) \in A$ and $u_b(\omega, x) \in V$ be the solution of (2) for $b(\omega, x) \in A$. Then, for all $v \in V$, we have

$$\mathbb{E}\left[\int_D a(\omega, x)\nabla u_a(\omega, x) \cdot \nabla v(\omega, x) dx\right] = \mathbb{E}\left[\int_D f(\omega, x) v(\omega, x)\, dx\right],$$
$$\mathbb{E}\left[\int_D b(\omega, x)\nabla u_b(\omega, x) \cdot \nabla v(\omega, x) dx\right] = \mathbb{E}\left[\int_D f(\omega, x) v(\omega, x)\, dx\right].$$

By simple algebraic manipulations, for a constant $c > 0$, we obtain

$$\|u_a(\omega, x) - u_b(\omega, x)\|_V \leq c\|a(\omega, x) - b(\omega, x)\|_{L^\infty(\Omega\times D)}.$$

$\square$

Differentiability of the parameter-to-solution map

Theorem

For each $a(\omega, x)$ in the interior of A, the map $a(\omega, x) \mapsto u_a(\omega, x)$ is differentiable at $a(\omega, x)$. The derivative $\delta u_a := Du_a(\delta a)$ of $u_a(\omega, x)$ at $a(\omega, x)$ in the direction $\delta a(\omega, x)$ is the unique solution of the stochastic variational problem: Find $\delta u_a(\omega, x) \in V$ such that

$$\mathbb{E}\left[\int_D a(\omega, x)\nabla\delta u_a(\omega, x)\cdot\nabla v(\omega, x)dx\right] = -\mathbb{E}\left[\int_D \delta a(\omega, x)\nabla u_a(\omega, x)\cdot\nabla v(\omega, x)\,dx\right],$$

for every $v(\omega, x) \in V$.

Two Optimization Formulations for the Inverse Problem

- The first one is commonly known output least-squares functional:

$$\widehat{J}_0(a) := \frac{1}{2}\mathbb{E}\left[\|u_a(\omega, x) - z(\omega, x)\|^2\right], \tag{3}$$

where $u_a(\omega, x)$ is the solution of (2) for $a(\omega, x)$, $z(\omega, x) \in L^2(\Omega; L^2(D))$ is the measured data, and $\|\cdot\|$ is a suitable norm, Three common choices are the $L^2(D)$-norm, the $H^1(D)$-norm, and the $H^1(D)$-seminorm.

- The second one is a new energy least-squares (ELS) functional:

$$J_0(a) = \frac{1}{2}\mathbb{E}\left[\int_D a(\omega, x)|\nabla(u_a(\omega, x) - z(\omega, x))|^2 dx\right], \tag{4}$$

where $u_a(\omega, x)$ is the solution of (2) for $a(\omega, x)$ and $z(\omega, x) \in L^2(\Omega; H_0^1(D))$ is the data.

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Convexity of the ELS Objective

Theorem

The ELS functional given in (4) is convex in the interior of the set A.

Proof.

The first derivative of the ELS functional reads:

$$DJ_0(a)(\delta a)$$
$$= -\frac{1}{2}\mathbb{E}\left[\int_D \delta a(\omega,x)\nabla(u_a(\omega,x)+z(\omega,x))\cdot\nabla(u_a(\omega,x)-z(\omega,x))dx\right].$$

Moreover, for all $a(\omega,x)$ in the interior of A, we have

$$D^2J_0(a)(\delta a,\delta a)\geq \alpha\|\delta u_a(\omega,x)\|_V^2.$$

□

Regularized ELS Objective

- For a stable identification process, inverse problems need some type of **regularization**.
- We consider the following admissible set:

$$A := \left\{ a \in H = L^2(\Omega; H(D)) : \ 0 < k_0 \le a(\omega, x) \le k_1 \text{ a.s. } \Omega \times D \right\},$$

where H is a separable Hilbert space compactly embedded into $B := L^\infty(\Omega; L^\infty(\Omega))$, and $H(D)$ is continuously embedded in $L^\infty(\Omega)$

- We consider the regularized energy least-squares functional

$$\min_{a \in A} J_\kappa(a) := \frac{1}{2}\mathbb{E}\left[\int_D a(\omega, x) |\nabla (u(\omega, x) - z(\omega, x))|^2 dx \right] + \frac{\kappa}{2} \|a(\omega, x)\|_H^2, \tag{5}$$

where $u_a(\omega, x)$ is the solution for $a(\omega, x)$, $z(\omega, x) \in L^2(\Omega; H_0^1(D))$ is the data, $\kappa > 0$ is a regularization parameter, and $\| \cdot \|_H^2$ is the regularizer.

Theorem

For each $\kappa > 0$, the ELS based problem (5) has a unique solution.

Finite-dimensional noise and Parametrization (I)

A vital component of the study of stochastic PDEs and stochastic optimization problems is the representation of the random fields by a finite number of random variables.

Definition

A function $v \in L^2(\Omega; L^2(D))$ of the form $v(x, \xi(\omega))$ for $x \in D$ and $\omega \in \Omega$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \Omega \mapsto \Gamma \subset \mathbb{R}^M$ and $\Gamma := \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_M$, is called a **finite-dimensional noise**. Here $\xi_k : \Omega \mapsto \Gamma_k$, for $k = 1, \ldots, M$, be real-valued random variables with $M < \infty$.

If a random field $v(x, \xi)$ is finite-dimensional noise, a change of variables can be made for evaluating expectations. For instance, denoting by σ, the joint density of ξ, we have

$$\|v\|^2_{L^2(\Omega; L^2(D))} = \mathbb{E}\left[\|v\|^2_{L^2(D)}\right] = \int_\Gamma \sigma(y) \|v(y, \cdot)\|^2_{L^2(D)} dy.$$

Consequently, by defining $y_k := \xi_k(\omega)$ and setting $y = (y_1, y_2, \ldots, y_M)$, we associate a random field $v(x, \xi)$ with a finite-dimensional noise by a function $v(x, y)$ in the weighted L^2 space

$$L^2_\sigma(\Gamma; L^2(D)) := \left\{ v : \Gamma \times D \to \mathbb{R} : \int_\Gamma \sigma(y) \|v(\cdot, y)\|^2_{L^2(D)} dy < \infty \right\}.$$

We assume that $a(\omega, x)$ and $f(\omega, x)$ are finite-dimensional noises and given by

$$a(\omega, x) = a_0(x) + \sum_{k=1}^{P} a_k(x) \xi_k(\omega),$$

$$f(\omega, x) = f_0(x) + \sum_{k=1}^{L} f_k(x) \xi_k(\omega),$$

where the real-valued functions a_k and f_k are uniformly bounded.
It follows from the Doob-Dynkin lemma that a solution of (2) is finite-dimensional noise and u is a function of ξ where $\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \Omega \mapsto \Gamma$ and $M := \max\{P, L\}$.

Finite-dimensional noise and Parametrization (III)

Then, the variational problem (2) reduces to the following parametric deterministic variational problem: Find $u(y,x) \in V_\sigma := L^2_\sigma(\Gamma; H^1_0(D))$ such that for every $v(y,x) \in V_\sigma$, we have

$$\int_\Gamma \sigma(y) \int_D a(y,x) \nabla u(y,x) \cdot \nabla v(y,x) dx\, dy = \int_\Gamma \sigma(y) \int_D f(y,x) v(y,x)\, dx\, dy. \tag{6}$$

Consider the finite-dimensional noise variants of the OLS and the ELS:

$$\min_{a \in A} \widehat{J}_0(a) := \frac{1}{2} \int_\Gamma \sigma(y) \int_D |\left(u_a(y,x) - z(y,x)\right)|^2 dx\, dy, \tag{7}$$

$$\min_{a \in A} J_0(a) := \frac{1}{2} \int_\Gamma \sigma(y) \int_D a(y,x) |\nabla \left(u_a(y,x) - z(y,x)\right)|^2 dx\, dy, \tag{8}$$

where $u_a(y,x)$ solves (6) for $a(y,x)$ and $z(y,x)$ is the finite-dimensional noise data.

Finite-dimensional noise Derivative Characterization

Theorem

Let a be in the interior of A. Then, the derivative $\delta u_a := Du_a(\delta a)$ of $u_a(y,x)$ at $a(y,x)$ in the direction $\delta a(y,x)$ is the unique solution of the following parameterized variational problem such that for every $v \in V_\sigma$, we have

$$\int_\Gamma \sigma(y) \int_D a(y,x)\nabla\delta u_a(y,x)\cdot\nabla v(y,x)dx\,dy$$
$$= -\int_\Gamma \sigma(y)\int_D \delta a(y,x)\nabla u_a(y,x)\cdot\nabla v(y,x)\,dx\,dy.$$

Furthermore, the derivative of the finite-dimensional noise ELS (8) reads:

$$DJ_0(a)(\delta a)$$
$$= \frac{1}{2}\int_\Gamma \sigma(y)\int_D \delta a(y,x)\nabla(u_a(y,x)+z(y,x))\cdot\nabla(u_a(y,x)-z(y,x))dx\,dy.$$

Stochastic Galerkin Based Computational Framework

Let V_{hk} be a finite-dimensional subspace of V_σ. An element $u_{hk} \in V_{hk}$ is the stochastic Galerkin solution if for all $v \in V_{hk}$:

$$\int_\Gamma \sigma(y) \int_D a(y,x) \nabla u_{hk}(y,x) \cdot \nabla v(y,x) dx\, dy = \int_\Gamma \sigma(y) \int_D f(y,x) v(y,x) dx\, dy.$$

Let V_h be an N-dimensional subspace of $H_0^1(D)$ and S_k be a Q-dimensional subspace of $L^2_\sigma(\Gamma)$ with

$$V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\},$$
$$S_k = \text{span}\{\psi_1, \psi_2, \ldots, \psi_Q\}.$$

We assume that the basis $\{\psi_1, \psi_2, \ldots, \psi_Q\}$ is orthonormal with respect to σ, that is,

$$\int_\Gamma \sigma(y) \psi_n(y) \psi_m(y) dy = \delta_{nm},$$

where δ_{nm} is the Kronecker delta: $\delta_{nm} = 1$ for $n = m$, $\delta_{nm} = 0$ for $n \neq m$.

Stochastic Galerkin. Finite Dimensional Spaces

We construct a finite-dimensional subspace of V_σ by tensorising the basis functions ϕ_i and ψ_j. That is, the following NQ-dimensional subspace will be the trial and test space for solving the discrete variational problem:

$$V_{hk} := V_h \otimes S_k := \operatorname{span}\{\phi_i \psi_j | \ i = 1, \dots, N, \ j = 1, \dots, Q\}.$$

Therefore, any $v \in V_h \otimes S_k$ has the representation

$$v(y,x) = \sum_{i=1}^{N} \sum_{j=1}^{Q} V_{ij}\phi_i(x)\psi_j(y) = \sum_{j=1}^{Q} \left[\sum_{i=1}^{N} V_{ij}\phi_i(x) \right] \psi_j(y) = \sum_{j=1}^{Q} V_j(x)\psi_j(y),$$

where

$$V_j(x) \equiv \sum_{i=1}^{N} V_{ij}\phi_i(x).$$

It is convenient to introduce the following vectorized notation

$$V = \begin{bmatrix} V_1 & V_2 & \dots & V_Q \end{bmatrix}^\top$$

where

$$V_j := \begin{bmatrix} V_{1j} & \dots & V_{Nj} \end{bmatrix}^\top \in \mathbb{R}^N.$$

Stochastic Galerkin. Coefficient Expansion

For the unknown random field we have assumed the linear expansion:

$$a(y,x) = a_0(x) + \sum_{s=1}^{M} y_s a_s(x) = \sum_{s=0}^{M} y_s a_s(x), \tag{9}$$

where, by convention, we denote $y_0 = 1$. The spatial components a_s are discretized by using another P-dimensional space

$$A_h = \text{span}\{\varphi_1, ..., \varphi_P\}.$$

By following the same vectorial notation, we have

$$a(y,x) = \sum_{i=1}^{P} A_{i0}\varphi_i(x) + \sum_{s=1}^{M} \left(\sum_{i=1}^{P} A_{is}\varphi_i(x) \right) y_s = \sum_{s=0}^{M} A_s y_s, \tag{10}$$

where the vectors $A_s(x) \equiv (A_{is}) \in \mathbb{R}^P$ for $s = 0 \ldots, M$,

$$A = \begin{bmatrix} A_0 & A_1 & \ldots & A_M \end{bmatrix}^\top \in \mathbb{R}^{P(M+1)\times 1}.$$

Stochastic Galerkin. The Discrete Variational Problem

The discrete variational problem seeks $u_{hk}(y,x) \in V_h \otimes S_Q$ such that

$$\int_\Gamma \sigma(y)\psi_n(y)\left(\int_D a(y,x)\nabla u_{hk}(y,x)\nabla\phi_i(x)dx\right)dy = \int_\Gamma \sigma(y)\psi_n(y)\left(\int_D f(y,x)\phi_i(x)dx\right)dy,$$

for every $i = 1,\dots,N,\ n = 1,\dots,Q$.

By using the representation $u_{hk} = \sum_{k=1}^{N}\sum_{m=1}^{Q} U_{km}\phi_k(x)\psi_m(y)$, we obtain

$$\left(K(A_0) + \sum_{s=1}^{M} g_{nn}^s K(A_s)\right)U_n + \sum_{m\neq n}\sum_{s=1}^{M} g_{nm}^s K(A_s)U_m = F_n, \text{ for every } n = 1,\dots,$$

where for $s \in \{0,\dots,M\}$, we define $K(A_s) \in \mathbb{R}^{n\times n}$ and $g_{nm}^s \in \mathbb{R}$ by

$$K(A_s)_{i,k} = \int_D A_s(x)\nabla\phi_k(x)\nabla\phi_i(x)dx,$$

$$g_{nm}^s = \int_\Gamma \sigma(y)\psi_n(y)\psi_m(y)y_s dy.$$

Now, for $s \in \{0,\dots,M\}$, we set $G^s = (g_{nm}^s) \in \mathbb{R}^{Q\times Q}$.

Stochastic Galerkin. The Discrete Variational Problem

Summarising, the discrete variational problem can be written as the following system:

$$\begin{pmatrix} K(A_0)+\sum\limits_{s=1}^{M} g_{11}^{s}K(A_s) & \sum\limits_{s=1}^{M} g_{12}^{s}K(A_s) & \cdots & \sum\limits_{s=1}^{M} g_{1Q}^{s}K(A_s) \\ \sum\limits_{s=1}^{M} g_{21}^{s}K(A_s) & K(A_0)+\sum\limits_{s=1}^{M} g_{22}^{s}K(A_s) & & \sum\limits_{s=1}^{M} g_{2Q}^{s}K(A_s) \\ \vdots & & \ddots & \vdots \\ \sum\limits_{s=1}^{M} g_{Q1}^{s}K(A_t) & \cdots & \cdots & K(A_0)+\sum\limits_{s=1}^{M} g_{QQ}^{s}K(A_s) \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_Q \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_Q \end{pmatrix}.$$

By using Kronecker product $\otimes$, we can express this system in a compact form

$$\left[\sum_{s=0}^{M} G^s \otimes K(A_s)\right] U = F. \tag{11}$$

Stochastic Galerkin. Discrete ELS

$$\min_{a\in A} J_\kappa(a) = \frac{1}{2}\int_\Gamma \sigma(y)\int_D a(y,x)\left|\nabla\left(u_a - z\right)\right|^2 dx\, dy$$
$$+ \frac{\kappa}{2}\int_\Gamma \sigma(y)\left\|a(y,x)\right\|^2_{H^1(D)}\, dy,$$

Then, the discrete version is given by

$$J_\kappa(A) = \frac{1}{2}(U-Z)^\top\left(\sum_{s=0}^{M} G^s \otimes K(A^s)\right)(U-Z)$$
$$+ \frac{\kappa}{2}\, A^\top\left(\Psi \otimes (Q_A + K_A)\right)A, \quad \text{where}$$

$$\Psi_{s,t} = \int_\Gamma \sigma(y)y_s y_t dy, \text{ for every } s,t = 0,\ldots,M,$$
$$(Q_A)_{i,j} = \int_D \varphi_j(x)\varphi_i(x)dx, \text{ for every } i,j = 1,\ldots,P,$$
$$(K_A)_{i,j} = \int_D \nabla\varphi_j(x)\nabla\varphi_i(x)dx, \text{ for every } i,j = 1,\ldots,P.$$

We also recall that the continuous derivative formula is given by

$$DJ_0(a)(b) = -\frac{1}{2}\int_\Gamma \sigma(y)\int_D b(y,x)\nabla(u+z)\cdot\nabla(u-z)dx\,dy + \kappa\int_\Gamma \sigma(y)\left\langle a(y,x), b\right\rangle_{H^1(D)},$$

which leads to

$$DJ_\kappa(A)(B) = -\frac{1}{2}(U+Z)^\top\left[\sum_{s=0}^{M} G^s\otimes K(B_s)\right](U-Z) + \kappa A^\top\left(\Psi\otimes(Q_A+K_A)\right)B.$$

To obtain an explicit formula for the gradient $\nabla J_\kappa(A)$, we use the notion of **adjoint stiffness matrix** $L(\cdot)\in\mathbb{R}^{N\times P}$, satisfying

$$L(V)B = K(B)V, \text{ for every } B\in\mathbb{R}^P,\ V\in\mathbb{R}^N.$$

The gradient formula then reads:

$$\nabla J_\kappa(A) = -\frac{1}{2}\left[\ \sum_{i,j=1}^{Q} g_{ij}^0(U_i+Z_i)^\top L(U_j-Z_j)\quad\ldots\sum_{i,j=1}^{Q} g_{ij}^M(U_i+Z_i)^\top L(U_j-Z_j)\ \right] + \kappa A^\top\left(\Psi\otimes(Q_A+K_A)\right).$$

Similar formulas can be given for the OLS functional.

A Numerical Example

- **Two degrees** of **stochasticity** example.
- For $D = (0,1)$ and for $Y_1(\omega), Y_2(\omega) \sim U[0,1]$ uniformly distributed over $[0,1]$, we define the random fields

$$\bar{a}(\omega, x) = 3 + x^2 + Y_1(\omega)\cos(\pi x) + Y_2(\omega)\sin(2\pi x),$$
$$\bar{u}(\omega, x) = x(1-x)Y_1(\omega),$$

  and compute the right-hand side f accordingly.
- Stochastic domain given by $\Gamma = [0,1] \times [0,1]$. Here

$$\sigma(y_1, y_2) = 1$$

  and orthornormal **Legendre polynomials** on $[0,1] \times [0,1]$ are defined as tensorial product of the one dimensional ones.

A Numerical Example (cont.)

We measure the expectation and the variance of the identification error via the (relative) error functional. For example, for the ELS objective functional, we estimate the identification error by the quantities:

- **Identification errors**

$$\varepsilon^M_{\text{mean}}(a) = \frac{\sqrt{\int_D (\mathbb{E}[a(\omega,x)] - \mathbb{E}[a_h^M(\omega,x)])^2 dx}}{\sqrt{\int_D \mathbb{E}\left[a(\omega,x)\right]^2 dx}}$$

$$\varepsilon^M_{\text{var}}(a) = \frac{\sqrt{\int_D (\mathrm{Var}[a(\omega,x)] - \mathrm{Var}[a_h^M(\omega,x)])^2 dx}}{\sqrt{\int_D \mathrm{Var}[a(\omega,x)]^2 dx}},$$

  where a_h^M is the estimated coefficient by the ELS approach. Similarly, we measure the simulated data error by the quantities:

- **Simulation errors**

$$\varepsilon^M_{\text{mean}}(u) = \frac{\sqrt{\int_D (\mathbb{E}[u(\omega,x)] - \mathbb{E}[u_h(a_h^M)(\omega,x)])^2 dx}}{\sqrt{\int_D \mathbb{E}\left[u(\omega,x)\right]^2 dx}}$$

A Numerical example. Identification and simulation errors

dim V_h	$\varepsilon^M_{\text{mean}}(a)$	$\varepsilon^M_{\text{var}}(a)$	$\varepsilon^M_{\text{mean}}(u)$	$\varepsilon^M_{\text{var}}(u)$	CPU time
50	4.5268e-03	4.0556e-02	2.6610e-05	4.4452e-06	4.43 s.
100	6.2400e-03	1.9205e-02	1.7895e-05	1.5850e-06	27.4 s.
150	6.3291e-03	2.8512e-02	1.8012e-05	1.5886e-06	95.3 s.
200	6.9930e-03	3.2212e-02	1.7258e-05	1.3371e-06	212.3 s.

Numerical errors ELS with $\kappa =$1e-06

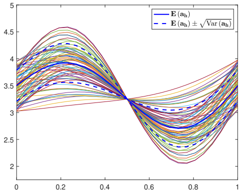
dim V_h	$\varepsilon^{HO}_{\text{mean}}(a)$	$\varepsilon^{HO}_{\text{var}}(a)$	$\varepsilon^{HO}_{\text{mean}}(u)$	$\varepsilon^{HO}_{\text{var}}(u)$	CPU time
50	1.1893e-02	3.7707e-02	5.5440e-05	3.6486e-06	6.11 s.
100	1.2621e-02	4.3191e-02	5.6433e-05	3.3801e-06	45.9 s.
150	1.4543e-02	6.5231e-02	7.2858e-05	4.7766e-06	126 s.
200	1.2355e-02	5.0199e-02	5.6855e-05	4.1495e-06	326 s.

Numerical errors H^1-OLS with $\kappa =$1e-06

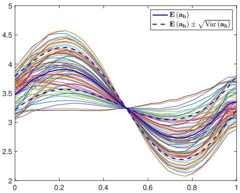
dim V_h	$\varepsilon^{LO}_{\text{mean}}(a)$	$\varepsilon^{LO}_{\text{var}}(a)$	$\varepsilon^{LO}_{\text{mean}}(u)$	$\varepsilon^{LO}_{\text{var}}(u)$	CPU time
50	7.7033e-02	3.9343e-01	8.9804e-04	5.3416e-05	4.98 s.
100	7.5412e-02	5.0431e-01	9.2701e-04	6.4441e-05	37.5 s.
150	7.6292e-02	4.3639e-01	9.1230e-04	5.8506e-05	151 s.
200	7.6757e-02	4.6106e-01	9.2280e-04	6.0029e-05	825 s.

Numerical errors L^2-OLS with $\kappa =$1e-06

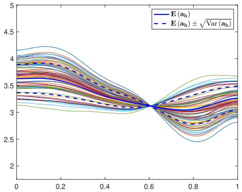
A Numerical example. Parameter identification



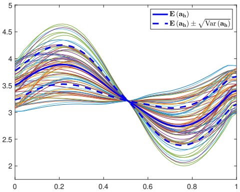
(a) Exact parameter a_h



(b) Estimated parameter a_h^{ELS}

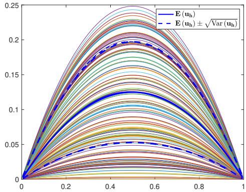


(c) Estimated parameter $a_h^{L^2-OLS}$

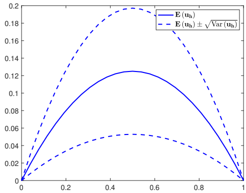


(d) Estimated parameter $a_h^{H^1-OLS}$

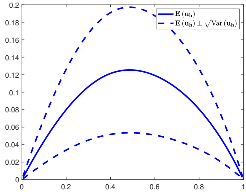
A Numerical example. Data Simulation



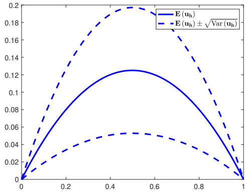
(a) Real data u (75 realizations)



(b) Simulated data $u_h(a_h^{ELS})$



(c) Simulated data $u_h(a_h^{L^2-OLS})$



(d) Simulated data $u_h(a_h^{H^1-OLS})$

Future Goals

- Identification of Stochastic Material Parameters
- Stochastic Gradient and Stochastic Extra-gradient for Inverse Problems

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